

# THE BOCHNER–HARTOGS DICHOTOMY FOR BOUNDED GEOMETRY HYPERBOLIC KÄHLER MANIFOLDS

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**ABSTRACT.** The main result is that for a connected hyperbolic complete Kähler manifold with bounded geometry of order two and exactly one end, either the first compactly supported cohomology with values in the structure sheaf vanishes or the manifold admits a proper holomorphic mapping onto a Riemann surface.

## INTRODUCTION

Let  $(X, g)$  be a connected noncompact complete Kähler manifold. According to [Gro1], [Li], [Gro2], [GroS], [NR1], [DelG], [NR5], and [NR6], if  $X$  has at least three filtered ends relative to the universal covering (i.e.,  $\tilde{e}(X) \geq 3$  in the sense of Definition 5.1) and  $X$  is weakly 1-complete (i.e.,  $X$  admits a continuous plurisubharmonic exhaustion function) or  $X$  is regular hyperbolic (i.e.,  $X$  admits a positive symmetric Green’s function that vanishes at infinity) or  $X$  has bounded geometry of order two (see Definition 2.1), then  $X$  admits a proper holomorphic mapping onto a Riemann surface. In particular, if  $X$  has at least three (standard) ends (i.e.,  $e(X) \geq 3$ ) and  $X$  satisfies one of the above three conditions, then such a mapping exists. Cousin’s example [Co] of a 2-ended weakly 1-complete covering of an Abelian variety that has only constant holomorphic functions demonstrates that two (filtered) ends do not suffice.

A noncompact complex manifold  $X$  for which  $H_c^1(X, \mathcal{O}) = 0$  is said to have the *Bochner–Hartogs property* (see Hartogs [Har], Bochner [Bo], and Harvey and Lawson [HavL]). Equivalently, for every  $\mathcal{C}^\infty$  compactly supported form  $\alpha$  of type  $(0, 1)$  with  $\bar{\partial}\alpha = 0$  on  $X$ , there is a  $\mathcal{C}^\infty$  compactly supported function  $\beta$  on  $X$  such that  $\bar{\partial}\beta = \alpha$ . If  $X$  has the Bochner–Hartogs property, then every holomorphic function on a neighborhood of infinity with no relatively compact connected components extends to a holomorphic function on  $X$ . For

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cutting off away from infinity, one gets a  $\mathcal{C}^\infty$  function  $\lambda$  on  $X$ . Taking  $\alpha \equiv \bar{\partial}\lambda$  and forming  $\beta$  as above, one then gets the desired extension  $\lambda - \beta$ . In particular,  $e(X) = 1$ , since for a complex manifold with multiple ends, there exists a locally constant function on a neighborhood of  $\infty$  that is equal to 1 along one end and 0 along the other ends, and such a function cannot extend holomorphically. Thus in a sense, the space  $H_c^1(X, \mathcal{O})$  is a function-theoretic approximation of the set of (topological) ends of  $X$ . An open Riemann surface  $S$ , as well as any complex manifold admitting a proper holomorphic mapping onto  $S$ , cannot have the Bochner–Hartogs property, because  $S$  admits meromorphic functions with finitely many poles. Examples of manifolds of dimension  $n$  having the Bochner–Hartogs property include strongly  $(n - 1)$ -complete complex manifolds (Andreotti and Vesentini [AnV]) and strongly hyper- $(n - 1)$ -convex Kähler manifolds (Grauert and Riemenschneider [GraR]). We will say that the *Bochner–Hartogs dichotomy* holds for a class of connected complex manifolds if each element either has the Bochner–Hartogs property or admits a proper holomorphic mapping onto a Riemann surface.

According to [Rs], [NR2], and [NR4], the Bochner–Hartogs dichotomy holds for the class of weakly 1-complete or regular hyperbolic complete Kähler manifolds with exactly one end. The main goal of this paper is the following:

**Theorem 0.1.** *Let  $X$  be a connected noncompact hyperbolic complete Kähler manifold with bounded geometry of order two, and assume that  $X$  has exactly one end. Then  $X$  admits a proper holomorphic mapping onto a Riemann surface if and only if  $H_c^1(X, \mathcal{O}) \neq 0$ .*

In other words, the Bochner–Hartogs dichotomy holds for the class of hyperbolic connected noncompact complete Kähler manifolds with bounded geometry of order two and exactly one end. When combined with the earlier results, the above gives the following:

**Corollary 0.2.** *Let  $X$  be a connected noncompact complete Kähler manifold that has exactly one end (or has at least three filtered ends) and satisfies at least one of the following:*

- (i)  *$X$  is weakly 1-complete;*
- (ii)  *$X$  is regular hyperbolic; or*
- (iii)  *$X$  is hyperbolic and of bounded geometry of order two.*

*Then  $X$  admits a proper holomorphic mapping onto a Riemann surface if and only if  $H_c^1(X, \mathcal{O}) \neq 0$ .*

In particular, since connected coverings of compact Kähler manifolds have bounded geometry of all orders, we have the following (cf. [ArBR], [Rs], and Theorem 0.2 of [NR4]):

**Corollary 0.3.** *Let  $X$  be a compact Kähler manifold, and  $\widehat{X} \rightarrow X$  a connected infinite covering that is hyperbolic and has exactly one end (or at least three filtered ends). Then  $\widehat{X}$  admits a proper holomorphic mapping onto a Riemann surface if and only if  $H_c^1(\widehat{X}, \mathcal{O}) \neq 0$ .*

A standard method for constructing a proper holomorphic mapping onto a Riemann surface is to produce suitable linearly independent holomorphic 1-forms (usually as holomorphic differentials of pluriharmonic functions), and to then apply versions of Gromov’s cup product lemma and the Castelnuovo–de Franchis theorem. In this context, an *irregular* hyperbolic manifold has a surprising advantage over a *regular* hyperbolic manifold in that an irregular hyperbolic complete Kähler manifold with bounded geometry of order two automatically admits a nonconstant positive pluriharmonic function. In particular, the proof of Theorem 0.1 in the irregular hyperbolic case is, in a sense, simpler than the proof in the regular hyperbolic case (which appeared in [NR4]). Because the existence of irregular hyperbolic complete Kähler manifolds with one end and bounded geometry of order two is not completely obvious, a 1-dimensional example is provided in Section 6. However, the authors do not know whether or not there exist examples with the above properties that satisfy the Bochner–Hartogs property (and hence do not admit proper holomorphic mappings onto Riemann surfaces).

Section 1 is a consideration of some elementary properties of ends, as well as some elementary topological properties of complex manifolds with the Bochner–Hartogs property. Section 2 contains the definition of bounded geometry. Section 3 consists of some terminology and facts from potential theory, and a proof that the Bochner–Hartogs property holds for any one-ended connected noncompact hyperbolic complete Kähler manifold with no nontrivial  $L^2$  holomorphic 1-forms. A modification of Nakai’s construction of an infinite-energy positive quasi-Dirichlet finite harmonic function on an irregular hyperbolic manifold, as well as a modification of a theorem of Sullivan which gives pluriharmonicity in the setting of a complete Kähler manifold with bounded geometry of order two, appear in Section 4. The proof of Theorem 0.1 and the proofs of some related results appear in Section 5. An example of an irregular hyperbolic complete Kähler manifold with one end and bounded geometry of all orders is constructed in Section 6.

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## 1. ENDS AND THE BOCHNER–HARTOGS PROPERTY

In this section, we consider an elementary topological property of complex manifolds with the Bochner–Hartogs property. Further topological characterizations of the Bochner–Hartogs dichotomy will be considered in Section 5. We first recall some terminology and facts concerning ends.

By an *end* of a connected manifold  $M$ , we will mean either a component  $E$  of  $M \setminus K$  with noncompact closure, where  $K$  is a given compact subset of  $M$ , or an element of

$$\lim_{\leftarrow} \pi_0(M \setminus K),$$

where the limit is taken as  $K$  ranges over the compact subsets of  $M$  (or equivalently, the compact subsets of  $M$  for which the complement  $M \setminus K$  has no relatively compact components, since the union of any compact subset of  $M$  with the relatively compact connected components of its complement is compact). The number of ends of  $M$  will be denoted by  $e(M)$ . For a compact set  $K$  such that  $M \setminus K$  has no relatively compact components, we will call

$$M \setminus K = E_1 \cup \cdots \cup E_m,$$

where  $\{E_j\}_{j=1}^m$  are the distinct components of  $M \setminus K$ , an *ends decomposition* for  $M$ .

**Lemma 1.1.** *Let  $M$  be a connected noncompact  $\mathcal{C}^\infty$  manifold.*

- (a) *If  $S \Subset M$ , then the number of components of  $M \setminus S$  that are not relatively compact in  $M$  is at most the number of components of  $M \setminus T$  for any set  $T$  with  $S \subset T \Subset M$ . In particular, the number of such components of  $M \setminus S$  is at most  $e(M)$ .*
- (b) *If  $K$  is a compact subset of  $M$ , then there exists a  $\mathcal{C}^\infty$  relatively compact domain  $\Omega$  in  $M$  containing  $K$  such that  $M \setminus \Omega$  has no compact components. In particular, if  $k$  is a positive integer with  $k \leq e(M)$ , then we may choose  $\Omega$  so that  $M \setminus \Omega$  also has at least  $k$  components; and hence  $\partial\Omega$  has at least  $k$  components.*
- (c) *If  $\Omega$  is a nonempty relatively compact domain in  $M$ , then the number of components of  $\partial\Omega$  is at most  $e(\Omega)$ , with equality if  $\Omega$  is also smooth.*

- (d) *Given an ends decomposition  $M \setminus K = E_1 \cup \cdots \cup E_m$ , there is a connected compact set  $K' \supset K$  such that any domain  $\Theta$  in  $M$  containing  $K'$  has an ends decomposition  $\Theta \setminus K = E'_1 \cup \cdots \cup E'_m$ , where  $E'_j = E_j \cap \Theta$  for  $j = 1, \dots, m$ .*
- (e) *If  $\Omega$  and  $\Theta$  are domains in  $M$  with  $\Theta \subset \Omega$  and both  $M \setminus \Omega$  and  $\Omega \setminus \Theta$  have no compact components, then  $M \setminus \Theta$  has no compact components.*
- (f) *If  $M$  admits a proper surjective continuous open mapping onto an orientable topological surface that is not simply connected, then there exists a  $\mathcal{C}^\infty$  relatively compact domain  $\Omega$  in  $M$  such that  $M \setminus \Omega$  has no compact components and  $\partial\Omega$  is not connected.*

*Proof.* For the proof of (a), we simply observe that if  $S \subset T \Subset M$ , then each connected component of  $M \setminus S$  that is not relatively compact in  $M$  must meet  $M \setminus T$  and must therefore contain some component of  $M \setminus T$ . Choosing  $T \supset S$  to be a compact set for which  $M \setminus T$  has no relatively compact components, we see that the number of components of  $M \setminus S$  is at most  $e(M)$ .

For the proof of (b), observe that given a compact set  $K \subset M$ , we may fix a  $\mathcal{C}^\infty$  domain  $\Omega_0$  with  $K \subset \Omega_0 \Subset M$ . The union of  $\Omega_0$  with those (finitely many) components of  $M \setminus \Omega_0$  which are compact is then a  $\mathcal{C}^\infty$  relatively compact domain  $\Omega \supset K$  in  $M$  for which  $M \setminus \Omega$  has no compact components. Given a positive integer  $k \leq e(M)$ , we may choose  $\Omega$  to also contain a compact set  $K'$  for which  $M \setminus K'$  has at least  $k$  components and no relatively compact components in  $M$ . Part (a) then implies that  $M \setminus \Omega$  has at least  $k$  components, and since each component must contain a component of  $\partial\Omega$ , we see also that  $\partial\Omega$  has at least  $k$  components.

For the proof of (c), suppose  $\Omega$  is a nonempty relatively compact domain in  $M$  and  $k$  is a positive integer. If  $\partial\Omega$  has at least  $k$  components, then we may fix a covering of  $\partial\Omega$  by disjoint relatively compact open subsets  $U_1, \dots, U_k$  of  $M$  each of which meets  $\partial\Omega$  (one may prove the existence of such sets by induction on  $k$ ). We may also fix a compact set  $K \subset \Omega$  containing  $\Omega \setminus (U_1 \cup \cdots \cup U_k)$  such that the components of  $\Omega \setminus K \subset U_1 \cup \cdots \cup U_k$  are not relatively compact in  $\Omega$ . For each  $j = 1, \dots, k$ ,  $U_j$  meets  $\partial\Omega$  and therefore some component  $E$  of  $\Omega \setminus K$ , and hence  $E \subset U_j$ . Since  $\Omega \setminus K$  has at most  $e(\Omega)$  components, it follows that  $k \leq e(\Omega)$ . Furthermore, if  $\Omega$  is smooth, then we may choose  $k$  to be equal to the number of boundary components, and we may choose the arbitrarily small neighborhoods so that  $U_j \cap \Omega$  is connected for each  $j$ . We then get  $k = e(\Omega)$  in this case.

For the proof of (d), let  $M \setminus K = E_1 \cup \cdots \cup E_m$  be an ends decomposition. We may fix a  $\mathcal{C}^\infty$  relatively compact domain  $\Omega$  in  $M$  containing  $K$  such that  $M \setminus \Omega$  has no compact components, and for each  $j = 1, \dots, m$ , we may fix a connected compact set  $A_j \subset E_j$  such that  $A_j$  meets each of the finitely many components of  $E_j \cap \partial\Omega$ . The compact set  $K' \equiv \overline{\Omega} \cup \bigcup_{j=1}^m A_j$  then has the required properties.

For the proof of (e), suppose  $\Omega$  and  $\Theta$  are domains in  $M$  with  $\Theta \subset \Omega$ , and both  $M \setminus \Omega$  and  $\Omega \setminus \Theta$  have no compact components. If  $E$  is a component of  $M \setminus \Theta$ , then either  $E$  meets  $M \setminus \Omega$ , in which case  $E$  contains a noncompact component of  $M \setminus \Omega$ , or  $E \subset \Omega$ , in which case  $E$  is a component of  $\Omega \setminus \Theta$ . In either case,  $E$  is noncompact.

Finally for the proof of (f), suppose  $\Phi: M \rightarrow S$  is a proper surjective continuous open mapping onto an orientable topological surface  $S$  that is not simply connected. By part (b), we may assume without loss of generality that  $e(M) = 1$ . If  $U$  is any open set in  $S$  and  $V$  is any component of  $\Phi^{-1}(U)$ , then  $\Phi(V)$  is both open and closed in  $U$ ; i.e.,  $\Phi(V)$  is a component of  $U$ . Consequently, if  $K \subset S$  is a compact set for which  $S \setminus K$  has no relatively compact components and  $V$  is any component of  $M \setminus \Phi^{-1}(K) = \Phi^{-1}(S \setminus K)$ , then  $\Phi(V)$  must be a component of  $S \setminus K$ . Hence  $V$  must be the unique component of  $M \setminus \Phi^{-1}(K)$  that is not relatively compact in  $M$ , and it follows that  $V = M \setminus \Phi^{-1}(K)$  and  $\Phi(V) = S \setminus K$  are connected. In particular,  $e(S) = 1$ .

Since every planar domain with one end is simply connected,  $S$  must be nonplanar; that is, there exists a nonseparating simple closed curve in  $S$ . Hence there exists a homeomorphism  $\Psi$  of a suitable annulus  $\Delta(0; r', R') \equiv \{z \in \mathbb{C} \mid r' < |z| < R'\}$  onto a domain  $A' \Subset S$  with connected complement  $S \setminus A'$ . Fixing  $r$  and  $R$  with  $0 < r' < r < R < R'$ , setting  $A \equiv \Psi(\Delta(0; r, R)) \Subset A'$ ,  $F \equiv S \setminus \overline{A}$  and  $E \equiv \Phi^{-1}(F) = M \setminus \Phi^{-1}(\overline{A})$ , and letting  $\Theta$  be a component of  $\Phi^{-1}(A)$ , we see that  $E$  is connected and  $\Phi(\Theta) = A$  (by the above), and that  $\overline{E} \subset M \setminus \Theta$ . It also follows that  $M \setminus \Theta$  is connected. For if  $K$  is a compact component of  $M \setminus \Theta$ , then we must have  $K \cap \overline{E} = \emptyset$ . Forming a connected neighborhood  $U$  of  $K$  in  $M \setminus \overline{E} \subset M \setminus E = \Phi^{-1}(\overline{A})$ , we get  $\Phi(K) \subset \Phi(U) \subset \overline{A}$ , and hence  $\Phi(U) \subset A$ . Thus  $K$  must lie in some component  $V \subset M \setminus \Theta$  of  $\Phi^{-1}(A)$ , and hence  $K = V$ . But then  $K$  must be both open and closed in  $M$ , which is clearly impossible. Therefore  $M \setminus \Theta$  is connected. Moreover, since  $\Phi(\Theta) = A$ , we must have  $\Phi(\partial\Theta) = \partial A$ , and hence  $\partial\Theta$  is not connected. Applying parts (b), (c), and (e), we get the desired smooth domain  $\Omega \Subset \Theta$ .  $\square$

As indicated in the introduction, a connected noncompact complex manifold with the Bochner–Hartogs property must have exactly one end and cannot admit a proper holomorphic mapping onto a Riemann surface. In fact, the following elementary observations suggest that complex manifolds with the Bochner–Hartogs property are very different topologically from those admitting proper holomorphic mappings onto Riemann surfaces:

**Proposition 1.2.** *Let  $X$  be a connected noncompact complex manifold.*

- (a) *Assume that  $H_c^1(X, \mathcal{O}) = 0$ . Then  $e(X) = 1$ . In fact, if  $\Omega$  is any nonempty domain in  $X$  for which each connected component of the complement  $X \setminus \Omega$  is noncompact, then  $e(\Omega) = 1$ . In particular, if  $\Omega$  is a relatively compact domain in  $X$  and  $X \setminus \Omega$  is connected, then  $\partial\Omega$  is connected. Moreover, every compact orientable  $\mathcal{C}^\infty$  hypersurface in  $X$  is the boundary of some smooth relatively compact domain in  $X$ .*
- (b) *If  $X$  admits a surjective proper continuous open mapping onto an orientable topological surface that is not simply connected (for example, if  $X$  admits a proper holomorphic mapping onto a Riemann surface other than the disk or the plane), then there exists a  $\mathcal{C}^\infty$  relatively compact domain  $\Omega$  in  $X$  such that  $X \setminus \Omega$  is connected but  $\partial\Omega$  is not connected (and  $e(\Omega) > 1$ ). In particular,  $H_c^1(X, \mathcal{O}) \neq 0$ .*

*Proof.* For the proof of (a), let us assume that  $H_c^1(X, \mathcal{O}) = 0$ . As argued in the introduction, we must then have  $e(X) = 1$ . Next, we show that any compact orientable  $\mathcal{C}^\infty$  hypersurface  $M$  in  $X$  is the boundary of some relatively compact  $\mathcal{C}^\infty$  domain in  $X$ . For we may fix a relatively compact connected neighborhood  $U$  of  $M$  in  $X$  such that  $U \setminus M$  has exactly two connected components,  $U_0$  and  $U_1$ . We may also fix a relatively compact neighborhood  $V$  of  $M$  in  $U$  and a  $\mathcal{C}^\infty$  function  $\lambda$  on  $X \setminus M$  such that  $\text{supp } \lambda \Subset U$ ,  $\lambda \equiv 0$  on  $U_0 \cap V$ , and  $\lambda \equiv 1$  on  $U_1 \cap V$ . Hence  $\bar{\partial}\lambda$  extends to a  $\bar{\partial}$ -closed  $\mathcal{C}^\infty$   $(0, 1)$ -form  $\alpha$  on  $X$  with compact support in  $U \setminus M$ , and since  $H_c^1(X, \mathcal{O}) = 0$ , we have  $\alpha = \bar{\partial}\beta$  for some  $\mathcal{C}^\infty$  compactly supported function  $\beta$  on  $X$ . The difference  $f \equiv \lambda - \beta$  is then a holomorphic function on  $X \setminus M$  that vanishes on some nonempty open subset. If  $X \setminus M$  is connected, then  $f \equiv 0$  on the entire set  $X \setminus M$ , and in particular, the restriction  $\beta|_V$  is a  $\mathcal{C}^\infty$  function that is equal to 1 on  $U_1 \cap V$ , 0 on  $U_0 \cap V$ . Since  $M = V \cap \partial U_0 = V \cap \partial U_1$ , we have arrived at a contradiction. Thus  $X \setminus M$  cannot be connected, and hence  $X \setminus M$  must have exactly two connected components, one containing  $U_0$  and the other containing  $U_1$ .

Since  $e(X) = 1$ , one of these connected components must be a relatively compact  $\mathcal{C}^\infty$  domain with boundary  $M$  in  $X$ . It follows that in particular, the boundary of any relatively compact  $\mathcal{C}^\infty$  domain in  $X$  with connected complement must be connected.

Next, suppose  $\Omega$  is an arbitrary nonempty domain for which  $X \setminus \Omega$  has no compact components. If  $e(\Omega) > 1$ , then part (b) of Lemma 1.1 provides a  $\mathcal{C}^\infty$  relatively compact domain  $\Theta$  in  $\Omega$  such that  $\Omega \setminus \Theta$  has no compact components and  $\partial\Theta$  is not connected, and hence part (e) implies that  $X \setminus \Theta$  has no compact components; i.e.,  $X \setminus \Theta$  is connected. However, as shown above, any smooth relatively compact domain in  $X$  with connected complement must have connected boundary. Thus we have arrived at a contradiction, and hence  $\Omega$  must have only one end. In particular, if  $\Omega \subseteq X$  (and  $X \setminus \Omega$  is connected), then by part (c) of Lemma 1.1,  $\partial\Omega$  must be connected.

Part (b) follows immediately from part (f) of Lemma 1.1.  $\square$

## 2. BOUNDED GEOMETRY

In this section, we recall the definition of bounded geometry and we fix some conventions. Let  $X$  be a complex manifold with almost complex structure  $J: TX \rightarrow TX$ . By a *Hermitian metric* on  $X$ , we will mean a Riemannian metric  $g$  on  $X$  such that  $g(Ju, Jv) = g(u, v)$  for every choice of real tangent vectors  $u, v \in T_p X$  with  $p \in X$ . We call  $(X, g)$  a *Hermitian manifold*. We will also denote by  $g$  the complex bilinear extension of  $g$  to the complexified tangent space  $(TX)_\mathbb{C}$ . The corresponding real  $(1, 1)$ -form  $\omega$  is given by  $(u, v) \mapsto \omega(u, v) \equiv g(Ju, v)$ . The corresponding Hermitian metric (in the sense of a smoothly varying family of Hermitian inner products) in the holomorphic tangent bundle  $T^{1,0}X$  is given by  $(u, v) \mapsto g(u, \bar{v})$ . Observe that with this convention, under the holomorphic vector bundle isomorphism  $(TX, J) \xrightarrow{\cong} T^{1,0}X$  given by  $u \mapsto \frac{1}{2}(u - iJu)$ , the pullback of this Hermitian metric to  $(TX, J)$  is given by  $(u, v) \mapsto \frac{1}{2}g(u, v) - \frac{i}{2}\omega(u, v)$ . In a slight abuse of notation, we will also denote the induced Hermitian metric in  $T^{1,0}X$ , as well as the induced Hermitian metric in  $\Lambda^r(TX)_\mathbb{C} \otimes \Lambda^s(T^*X)_\mathbb{C}$ , by  $g$ . The corresponding Laplacians are given by:

$$\begin{aligned}\Delta &= \Delta_d \equiv -(dd^* + d^*d), \\ \Delta_{\bar{\partial}} &= -(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}), \\ \Delta_{\partial} &= -(\partial\partial^* + \partial^*\partial).\end{aligned}$$



If  $(X, g, \omega)$  is *Kähler*, i.e.,  $d\omega = 0$ , then  $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$ .

**Definition 2.1.** For  $S \subset X$  and  $k$  a nonnegative integer, we will say that a Hermitian manifold  $(X, g)$  of dimension  $n$  has *bounded geometry of order  $k$  along  $S$*  if for some constant  $C > 0$  and for every point  $p \in S$ , there is a biholomorphism  $\Psi$  of the unit ball  $B \equiv B_{g_{\mathbb{C}^n}}(0; 1) \subset \mathbb{C}^n$  onto a neighborhood of  $p$  in  $X$  such that  $\Psi(0) = p$  and such that on  $B$ ,

$$C^{-1}g_{\mathbb{C}^n} \leq \Psi^*g \leq Cg_{\mathbb{C}^n} \quad \text{and} \quad |D^m \Psi^*g| \leq C \text{ for } m = 0, 1, 2, \dots, k.$$

### 3. GREEN'S FUNCTIONS AND HARMONIC PROJECTIONS

In this section we recall some terminology and facts from potential theory (a more detailed outline is provided in [NR1]). We will also see that the Bochner–Hartogs property holds for a connected noncompact complete Kähler manifold with exactly one end and no nontrivial  $L^2$  holomorphic 1-forms.

A connected noncompact oriented Riemannian manifold  $(M, g)$  is called *hyperbolic* if there exists a positive symmetric Green's function  $G(x, y)$  on  $M$ ; otherwise,  $M$  is called *parabolic*. Equivalently,  $M$  is hyperbolic if given a relatively compact  $\mathcal{C}^\infty$  domain  $\Omega$  for which no connected component of  $M \setminus \Omega$  is compact, there is a connected component  $E$  of  $M \setminus \overline{\Omega}$  and a (unique) greatest  $\mathcal{C}^\infty$  function  $u_E : \overline{E} \rightarrow [0, 1)$  such that  $u_E$  is harmonic on  $E$ ,  $u_E = 0$  on  $\partial E$ , and  $\sup_E u_E = 1$  (see, for example, Theorem 3 of [GK]). We will also call  $E$ , and any end containing  $E$ , a *hyperbolic* end. An end that is not hyperbolic is called *parabolic*, and we set  $u_E \equiv 0$  for any parabolic end component  $E$  of  $M \setminus \Omega$ . We call the function  $u : M \setminus \Omega \rightarrow [0, 1)$  defined by  $u|_{\overline{E}} = u_E$  for each connected component  $E$  of  $M \setminus \overline{\Omega}$ , the *harmonic measure of the ideal boundary of  $M$  with respect to  $M \setminus \overline{\Omega}$* . A sequence  $\{x_\nu\}$  in  $M$  with  $x_\nu \rightarrow \infty$  and  $G(\cdot, x_\nu) \rightarrow 0$  (equivalently,  $u(x_\nu) \rightarrow 1$ ) is called a *regular sequence*. Such a sequence always exists (for  $M$  hyperbolic). A sequence  $\{x_\nu\}$  tending to infinity with  $\liminf_{\nu \rightarrow \infty} G(\cdot, x_\nu) > 0$  (i.e.,  $\limsup_{\nu \rightarrow \infty} u(x_\nu) < 1$  or equivalently,  $\{x_\nu\}$  has no regular subsequences) is called an *irregular sequence*. Clearly, every sequence tending to infinity that is not regular admits an irregular subsequence. We say that an end  $E$  of  $M$  is *regular* (*irregular*) if every sequence in  $E$  tending to infinity in  $M$  is regular (respectively, there exists an irregular sequence in  $E$ ). Another characterization of hyperbolicity is that  $M$  is hyperbolic if and only if  $M$  admits a nonconstant negative continuous subharmonic function  $\varphi$ . In fact, if  $\{x_\nu\}$  is a sequence in  $M$  with  $x_\nu \rightarrow \infty$  and  $\varphi(x_\nu) \rightarrow 0$ , then  $\{x_\nu\}$  is a regular sequence.

We recall that the *energy* (or *Dirichlet integral*) of a suitable function  $\varphi$  (for example, a function with first-order distributional derivatives) on a Riemannian manifold  $M$  is given by  $\int_M |\nabla \varphi|^2 dV$ . To any  $\mathcal{C}^\infty$  compactly supported  $\bar{\partial}$ -closed  $(0, 1)$ -form  $\alpha$  on a connected noncompact hyperbolic complete Kähler manifold  $X$ , we may associate a bounded finite-energy (i.e., Dirichlet-finite) pluriharmonic function on  $X \setminus \text{supp } \alpha$  that vanishes at infinity along any regular sequence:

**Lemma 3.1** (see, for example, Lemma 1.1 of [NR4]). *Let  $X$  be a connected noncompact complete hyperbolic Kähler manifold, and let  $\alpha$  be a  $\mathcal{C}^\infty$  compactly supported form of type  $(0, 1)$  on  $X$  with  $\bar{\partial}\alpha = 0$ . Then there exist a closed and coclosed  $L^2$  harmonic form  $\gamma$  of type  $(0, 1)$  and a  $\mathcal{C}^\infty$  bounded function  $\beta: X \rightarrow \mathbb{C}$  with finite energy such that  $\gamma = \alpha - \bar{\partial}\beta$  and  $\beta(x_\nu) \rightarrow 0$  for every regular sequence  $\{x_\nu\}$  in  $X$ .*

*Remarks.* 1. In particular,  $\bar{\gamma}$  is a holomorphic 1-form on  $X$ , and  $\beta$  is pluriharmonic on the complement of the support of  $\alpha$ .

2. Under certain conditions, the leaves of the foliation determined by  $\bar{\gamma}$  outside a large compact subset of  $X$  are compact, and one gets a proper holomorphic mapping onto a Riemann surface.

3. According to Lemma 3.2 below (which is a modification of an observation due to J. Wang), if  $\beta$  is holomorphic on some hyperbolic end, then  $\beta$  vanishes on that end.

**Lemma 3.2** (cf. Lemma 1.3 of [NR4]). *Let  $X$  be a connected noncompact complete (hyperbolic) Kähler manifold, and let  $E$  be a hyperbolic end of  $X$ . If  $f$  is a bounded holomorphic function on  $E$  and  $f(x_\nu) \rightarrow 0$  for every regular sequence  $\{x_\nu\}$  for  $X$  in  $E$ , then  $f \equiv 0$  on  $E$ .*

*Proof.* We may fix a nonempty smooth domain  $\Omega$  such that  $\partial E \subset \Omega \Subset X$  and  $X \setminus \Omega$  has no compact connected components. In particular, some component  $E_0$  of  $E \setminus \bar{\Omega}$  is a hyperbolic end of  $X$ . The harmonic measure of the ideal boundary of  $X$  with respect to  $X \setminus \bar{\Omega}$  is a nonconstant function  $u: X \setminus \Omega \rightarrow [0, 1)$ . By replacing  $f$  with the product of  $f$  and a sufficiently small nonzero constant, we may assume that  $|f| < 1$  and hence for each  $\epsilon > 0$ ,  $u + \epsilon \log |f| < 0$  on  $E \cap \partial\Omega$ . Thus we get a nonnegative bounded continuous subharmonic function  $\varphi_\epsilon$  on  $X$  by setting  $\varphi_\epsilon \equiv 0$  on  $X \setminus E_0$  and  $\varphi_\epsilon \equiv \max(0, u + \epsilon \log |f|)$  on  $E_0$ . If  $f(p) \neq 0$  at some point  $p \in E_0$ , then  $\varphi_\epsilon(p) > 0$  for  $\epsilon$  sufficiently small. However, any sequence  $\{x_\nu\}$  in  $E_0$  with  $\varphi_\epsilon(x_\nu) \rightarrow m \equiv \sup \varphi_\epsilon > 0$  must be a regular sequence and

must therefore satisfy  $u(x_\nu) + \epsilon \log |f(x_\nu)| \rightarrow -\infty$ , which contradicts the choice of  $\{x_\nu\}$ . Thus  $f$  vanishes on  $E_0$  and therefore, on  $E$ .  $\square$

The above considerations lead to the following observation (cf. Proposition 4.4 of [NR3]):

**Theorem 3.3.** *Let  $X$  be a connected noncompact hyperbolic complete Kähler manifold with no nontrivial  $L^2$  holomorphic 1-forms.*

- (a) *For every compactly supported  $\bar{\partial}$ -closed  $\mathcal{C}^\infty$  form  $\alpha$  of type  $(0, 1)$  on  $X$ , there exists a bounded  $\mathcal{C}^\infty$  function  $\beta$  with finite energy on  $X$  such that  $\bar{\partial}\beta = \alpha$  on  $X$  and  $\beta$  vanishes on every hyperbolic end  $E$  of  $X$  that is contained in  $X \setminus \text{supp } \alpha$ .*
- (b) *In any ends decomposition  $X \setminus K = E_1 \cup \cdots \cup E_m$ , exactly one of the ends, say  $E_1$ , is hyperbolic, and moreover, every holomorphic function on  $E_1$  admits a (unique) extension to a holomorphic function on  $X$ .*
- (c) *If  $e(X) = 1$  (equivalently, every end of  $X$  is hyperbolic), then  $H_c^1(X, \mathcal{O}) = 0$ .*

*Proof.* Given a compactly supported  $\bar{\partial}$ -closed  $\mathcal{C}^\infty$  form  $\alpha$  of type  $(0, 1)$  on  $X$ , Lemma 3.1 provides a bounded  $\mathcal{C}^\infty$  function  $\beta$  with finite energy such that  $\bar{\partial}\beta = \alpha$  and  $\beta(x_\nu) \rightarrow 0$  for every regular sequence  $\{x_\nu\}$  in  $X$  (by hypothesis, the  $L^2$  holomorphic 1-form  $\bar{\gamma}$  provided by the lemma must be trivial). In particular,  $\beta$  is holomorphic on  $X \setminus \text{supp } \alpha$ , and Lemma 3.2 implies that  $\beta$  must vanish on every hyperbolic end of  $X$  contained in  $X \setminus \text{supp } \alpha$ . Thus part (a) is proved.

For the proof of part (b), suppose  $X \setminus K = E_1 \cup \cdots \cup E_m$  is an ends decomposition. Then at least one of the ends, say  $E_1$ , must be hyperbolic. Given a function  $f \in \mathcal{O}(X \setminus K)$ , we may fix a relatively compact neighborhood  $U$  of  $K$  in  $X$  and a  $\mathcal{C}^\infty$  function  $\lambda$  on  $X$  such that  $\lambda \equiv f$  on  $X \setminus U$ . Applying part (a) to the  $(0, 1)$ -form  $\alpha \equiv \bar{\partial}\lambda$ , we get a  $\mathcal{C}^\infty$  function  $\beta$  such that  $\bar{\partial}\beta = \alpha$  on  $X$  and  $\beta \equiv 0$  on any hyperbolic end contained in  $X \setminus U$ . If  $E_j$  is a hyperbolic end (for example, if  $j = 1$ ), then  $E_j \setminus U$  must contain a hyperbolic end  $E$  of  $X$ , and the holomorphic function  $h \equiv \lambda - \beta$  on  $X$  must agree with  $f$  on  $E$  and therefore, on  $E_j$ . Thus we get a holomorphic function on  $X$  that agrees with  $f$  on every  $E_j$  which is hyperbolic. Taking  $f$  to be a locally constant function on  $X \setminus K$  with distinct values on the components  $E_1, \dots, E_m$ , we see that in fact,  $E_j$  must be a parabolic end for  $j = 2, \dots, m$ .

Part (c) follows immediately from parts (a) and (b).  $\square$

We close this section with a preliminary step toward the proof of Theorem 0.1:

**Lemma 3.4.** *Suppose  $(X, g)$  is a connected noncompact hyperbolic complete Kähler manifold with bounded geometry of order 0,  $e(X) = 1$ , and there exists a real-valued pluriharmonic function  $\rho$  with bounded gradient and infinite energy on  $X$ . Then  $X$  admits a proper holomorphic mapping onto a Riemann surface if and only if  $H_c^1(X, \mathcal{O}) \neq 0$ .*

*Proof.* Given a compactly supported  $\bar{\partial}$ -closed  $\mathcal{C}^\infty$  form  $\alpha$  of type  $(0, 1)$  on  $X$ , Lemma 3.1 provides a closed and coclosed  $L^2$  harmonic form  $\gamma$  of type  $(0, 1)$  and a  $\mathcal{C}^\infty$  bounded function  $\beta: X \rightarrow \mathbb{C}$  with finite energy such that  $\gamma = \alpha - \bar{\partial}\beta$  and  $\beta(x_\nu) \rightarrow 0$  for every regular sequence  $\{x_\nu\}$  in  $X$ . If  $\gamma \equiv 0$ , then  $\bar{\partial}\beta = \alpha$  and Lemma 3.2 implies that  $\beta$  vanishes on the complement of some compact set. If  $\gamma$  is nontrivial, then the  $L^2$  holomorphic 1-form  $\theta_1 \equiv \bar{\gamma}$  and the bounded holomorphic 1-form  $\theta_2 \equiv \partial\rho$ , which is not in  $L^2$ , must be linearly independent. Theorem 0.1 and Theorem 0.2 of [NR6] then provide a proper holomorphic mapping of  $X$  onto a Riemann surface.  $\square$

*Remark.* The proofs of Lemma 1.1 of [NR6] and Theorem 0.1 of [NR6] (the latter fact was applied above and relies on the former) contain a minor mistake in their application of continuity of intersections (see [Ste] or [TW] or Theorem 4.23 in [ABCKT]). In each of these proofs, one has a sequence of levels  $\{L_\nu\}$  of a holomorphic mapping  $f: X \rightarrow \mathbb{P}^1$  and a sequence of points  $\{x_\nu\}$  such that  $x_\nu \in L_\nu$  for each  $\nu$  and  $x_\nu \rightarrow p$ . For  $L$  the level of  $f$  through  $p$ , by continuity of intersections,  $\{L_\nu\}$  converges to  $L$  relative to the ambient manifold  $X \setminus [f^{-1}(f(p)) \setminus L]$ , but contrary to what was stated in these proofs, a priori, this convergence need not hold relative to  $X$ . Aside from this small misstatement, the proofs are correct and no further changes are needed.

#### 4. QUASI-DIRICHLET-FINITE PLURIHARMONIC FUNCTIONS

The following is the main advantage of working with *irregular* hyperbolic manifolds:

**Lemma 4.1** (Nakai). *Let  $(M, g)$  be a connected noncompact irregular hyperbolic oriented complete Riemannian manifold, let  $\{q_k\}$  be an irregular sequence, let  $G(\cdot, \cdot)$  be the Green's function, and let  $\rho_k \equiv G(\cdot, q_k): M \rightarrow (0, \infty]$  for each  $k$ . Then some subsequence of  $\{\rho_k\}$  converges uniformly on compact subsets of  $M$  to a function  $\rho$ . Moreover, any such limit function  $\rho$  has the following properties:*

- (i) *The function  $\rho$  is positive and harmonic;*
- (ii)  $\int_M |\nabla \rho|^2 dV_g = \infty$ ;

- (iii)  $\int_{\rho^{-1}([a,b])} |\nabla \rho|_g^2 dV_g \leq b - a$  for all  $a$  and  $b$  with  $0 \leq a < b$  (in particular,  $\rho$  is unbounded); and
- (iv) If  $\Omega$  is any smooth domain with compact boundary (i.e., either  $\Omega$  is an end or  $\Omega \Subset M$ ) and at most finitely many terms of the sequence  $\{q_k\}$  lie in  $\Omega$ , then

$$\sup_{\Omega} \rho = \max_{\partial\Omega} \rho < \infty \quad \text{and} \quad \int_{\Omega} |\nabla \rho|^2 dV \leq \int_{\partial\Omega} \rho \frac{\partial \rho}{\partial \nu} d\sigma < \infty.$$

*Remark.* Following Nakai [Na] and Sario and Nakai [SaNa], a positive function  $\varphi$  on a Riemannian manifold  $(M, g)$  is called *quasi-Dirichlet-finite* if there is a positive constant  $C$  such that

$$\int_{\varphi^{-1}([0,b])} |\nabla \varphi|_g^2 dV_g \leq Cb$$

for every  $b > 0$ . Nakai proved the existence of an Evans-type quasi-Dirichlet-finite positive harmonic function on an irregular Riemann surface. His arguments, which involve the behavior of the Green's function at the Royden boundary, carry over to a Riemannian manifold and actually show that the constructed function has the slightly stronger property appearing in the above lemma. One can instead prove the lemma via Nakai's arguments simply by taking  $\rho = G(\cdot, q)$ , where  $G$  is the extension of the Green's function to the Royden compactification and  $q$  is a point in the Royden boundary for which  $\rho > 0$  on  $M$ . The direct proof appearing below is essentially this latter argument.

*Proof of Lemma 4.1.* Fixing a sequence of nonempty smooth domains  $\{\Omega_m\}_{m=0}^{\infty}$  such that  $M \setminus \Omega_0$  has no compact connected components,  $\bigcup_{m=0}^{\infty} \Omega_m = M$ , and  $\Omega_{m-1} \Subset \Omega_m$  for  $m = 1, 2, 3, \dots$ , and letting  $G_m$  be the Green's function on  $\Omega_m$  for each  $m$ , we get  $G_m \nearrow G$ . Given  $m_0 \in \mathbb{Z}_{>0}$ , for each integer  $m > m_0$  and each point  $p \in \Omega_{m_0}$ , the continuous function  $G_m(p, \cdot)|_{\overline{\Omega}_m \setminus \Omega_{m_0}}$  vanishes on  $\partial\Omega_m$ , and the function is positive on  $\partial\Omega_{m_0}$  and harmonic on  $\Omega_m \setminus \overline{\Omega}_{m_0}$ . Thus

$$G_m(p, \cdot)|_{\overline{\Omega}_m \setminus \Omega_{m_0}} \leq \max_{\partial\Omega_{m_0}} G_m(p, \cdot) \leq \max_{\partial\Omega_{m_0}} G(p, \cdot).$$

Passing to the limit we get

$$G(p, \cdot) \leq \max_{\partial\Omega_{m_0}} G(p, \cdot)$$

on  $M \setminus \Omega_{m_0}$  for each point  $p \in \Omega_{m_0}$ . Hence

$$G \leq A_{m_0} \equiv \max_{\overline{\Omega}_{m_0-1} \times \partial\Omega_{m_0}} G$$

on  $\overline{\Omega}_{m_0-1} \times (M \setminus \Omega_{m_0})$ . In particular,  $\rho_k = G(\cdot, q_k) \leq A_{m_0}$  on  $\overline{\Omega}_{m_0-1}$  for  $k \gg 0$ . Therefore, by replacing  $\{q_k\}$  with a suitable subsequence, we may assume that  $\rho_k$  converges uniformly on compact subsets of  $M$  to a positive harmonic function  $\rho$ .

Suppose  $0 < a < b$ . Given  $k \in \mathbb{Z}_{>0}$ , for  $m \gg 0$  we have  $q_k \in \Omega_m$ , and the function  $\rho_k^{(m)} \equiv G_m(\cdot, q_k) : \overline{\Omega}_m \rightarrow [0, \infty]$  satisfies  $(\rho_k^{(m)})^{-1}((a, \infty)) \Subset \Omega_m$ . Hence if  $r$  and  $s$  are regular values of  $\rho_k^{(m)}|_{\Omega_m \setminus \{q_k\}}$  with  $a < r < s < b$ , then

$$\begin{aligned} \int_{(\rho_k^{(m)})^{-1}((r,s))} |\nabla \rho_k^{(m)}|^2 dV &= \int_{(\rho_k^{(m)})^{-1}(s)} \rho_k^{(m)} \frac{\partial \rho_k^{(m)}}{\partial \nu} d\sigma - \int_{(\rho_k^{(m)})^{-1}(r)} \rho_k^{(m)} \frac{\partial \rho_k^{(m)}}{\partial \nu} d\sigma \\ &= \int_{(\rho_k^{(m)})^{-1}(s)} s \cdot \frac{\partial \rho_k^{(m)}}{\partial \nu} d\sigma - \int_{(\rho_k^{(m)})^{-1}(r)} r \cdot \frac{\partial \rho_k^{(m)}}{\partial \nu} d\sigma \\ &= (r - s) \int_{\partial \Omega_m} \frac{\partial \rho_k^{(m)}}{\partial \nu} d\sigma \\ &= (s - r) \int_{\partial \Omega_m} (-1) \frac{\partial}{\partial \nu} [G_m(\cdot, q_k)] d\sigma \\ &= s - r, \end{aligned}$$

where  $\partial/\partial \nu$  is the normal derivative oriented outward for the open sets  $\Omega_m$ ,  $(\rho_k^{(m)})^{-1}((0, s))$ , and  $(\rho_k^{(m)})^{-1}((0, r))$ . Here we have normalized  $G_m$  (and similarly, all Green's functions) so that  $-\Delta_{\text{distr}} G_m(\cdot, q)$  is the Dirac function at  $q$  for each point  $q \in \Omega_m$ . Letting  $r \rightarrow a^+$  and  $s \rightarrow b^-$ , we get

$$\int_{(\rho_k^{(m)})^{-1}((a,b))} |\nabla \rho_k^{(m)}|^2 dV = (b - a).$$

Letting  $\chi_A$  denote the characteristic function of each set  $A \subset M$ , we have

$$\lim_{m \rightarrow \infty} |\nabla \rho_k^{(m)}| = |\nabla \rho_k| \text{ on } M \setminus \{q_k\} \quad \text{and} \quad \liminf_{m \rightarrow \infty} \chi_{(\rho_k^{(m)})^{-1}((a,b))} \geq \chi_{\rho_k^{-1}((a,b))}.$$

Hence Fatou's lemma gives

$$\int_{\rho_k^{-1}((a,b))} |\nabla \rho_k|^2 dV \leq (b - a).$$

Similarly, letting  $k \rightarrow \infty$ , we get

$$\int_{\rho^{-1}((a,b))} |\nabla \rho|^2 dV \leq (b - a).$$

Applying this inequality to  $a'$  and  $b'$  with  $0 < a' < b'$  and letting  $a' \rightarrow a^-$  and  $b' \rightarrow b^+$ , we get

$$\int_{\rho^{-1}([a,b])} |\nabla \rho|^2 dV \leq (b-a).$$

Letting  $a \rightarrow 0^+$  (and noting that  $\rho > 0$ ), we also get the above inequality for  $a = 0$ .

Assuming now that  $\rho$  has finite energy, we will reason to a contradiction. We may fix a constant  $b > \sup_{\Omega_0} \rho$  that is a regular value of  $\rho$ , of  $\rho_k|_{M \setminus \{q_k\}}$  for all  $k$ , and of  $\rho_k^{(m)}|_{\Omega_m \setminus \{q_k\}}$  for all  $k$  and  $m$ . Note that we have not yet shown that  $\rho$  is unbounded, so we have not yet ruled out the possibility that  $\rho^{-1}((0,b)) = M$ , and in particular, that  $\rho^{-1}(b) = \emptyset$ . Since  $\rho_k \rightarrow \rho$  uniformly on compact subsets of  $M$  as  $k \rightarrow \infty$ , and for each  $k$ ,  $\rho_k^{(m)} \rightarrow \rho_k$  uniformly on compact subsets of  $M \setminus \{q_k\}$  as  $m \rightarrow \infty$ , we may fix a positive integer  $k_0$  and a strictly increasing sequence of positive integers  $\{m_k\}$  such that  $q_k \in \Omega_{m_k}$  for each  $k$ ,  $\rho_k^{(m_k)} \leq \rho_k < b$  on  $\overline{\Omega}_0$  for each  $k \geq k_0$ , and  $\rho_k^{(m_k)} \rightarrow \rho$  uniformly on compact sets as  $k \rightarrow \infty$ . Letting  $\varphi \equiv \min(\rho, b)$  and letting  $\varphi_k: M \rightarrow [0, b]$  be the Lipschitz function given by

$$\varphi_k \equiv \begin{cases} \min(\rho_k^{(m_k)}, b) & \text{on } \overline{\Omega}_{m_k} \\ 0 & \text{elsewhere} \end{cases}$$

for each  $k$ , we see that  $\varphi_k \rightarrow \varphi$  uniformly on compact subsets of  $M$  and  $\nabla \varphi_k \rightarrow \nabla \varphi$  uniformly on compact subsets of  $M \setminus \rho^{-1}(b)$ . Moreover, for each  $k$ ,

$$\int_M |\nabla \varphi_k|^2 dV = \int_{(\rho_k^{(m_k)})^{-1}((0,b))} |\nabla \rho_k^{(m_k)}|^2 dV = b.$$

Applying weak compactness, we may assume that  $\{\nabla \varphi_k\}$  converges weakly in  $L^2$  to a vector field  $v$ . But for each compact set  $K \subset M \setminus \rho^{-1}(b)$ ,  $(\nabla \varphi_k)|_K \rightarrow (\nabla \varphi)|_K$  uniformly, and therefore in  $L^2$ . Since  $\rho^{-1}(b)$  is a set of measure 0, we must have  $v = \nabla \varphi$  (in  $L^2$ ). Hence

$$\begin{aligned} \int_{\rho^{-1}((0,b))} |\nabla \rho|^2 dV &= \langle \nabla \varphi, \nabla \rho \rangle \leftarrow \langle \nabla \varphi_k, \nabla \rho \rangle = \int_{\partial \Omega_{m_k}} \rho_k^{(m_k)} \frac{\partial \rho}{\partial \nu} d\sigma \\ &\quad + \int_{(\rho_k^{(m_k)})^{-1}(b)} \rho_k^{(m_k)} \frac{\partial \rho}{\partial \nu} d\sigma \\ &= 0 - b \int_{\partial((\rho_k^{(m_k)})^{-1}((b,\infty]))} \frac{\partial \rho}{\partial \nu} d\sigma = 0. \end{aligned}$$

It follows that  $\rho \equiv a$  for some constant  $a$  (in particular,  $0 < a < b$ ). Letting  $u$  be the harmonic measure of the ideal boundary of  $M$  with respect to  $M \setminus \overline{\Omega}_0$  and letting

$\psi: M \rightarrow [0, 1)$  be the Dirichlet-finite locally Lipschitz function on  $M$  obtained by extending  $u$  by 0, we get

$$\begin{aligned}
0 = \langle 0, \nabla \psi \rangle &\leftarrow \langle \nabla \varphi_k, \nabla \psi \rangle = \int_{\partial \Omega_{m_k}} \rho_k^{(m_k)} \frac{\partial u}{\partial \nu} d\sigma - \int_{\partial \Omega_0} \rho_k^{(m_k)} \frac{\partial u}{\partial \nu} d\sigma \\
&\quad + \int_{(\rho_k^{(m_k)})^{-1}(b)} \rho_k^{(m_k)} \frac{\partial u}{\partial \nu} d\sigma \\
&= 0 - \int_{\partial \Omega_0} \rho_k^{(m_k)} \frac{\partial u}{\partial \nu} d\sigma - b \int_{\partial((\rho_k^{(m_k)})^{-1}((b, \infty]))} \frac{\partial u}{\partial \nu} d\sigma \\
&= - \int_{\partial \Omega_0} \rho_k^{(m_k)} \frac{\partial u}{\partial \nu} d\sigma \rightarrow -a \int_{\partial \Omega_0} \frac{\partial u}{\partial \nu} d\sigma < 0.
\end{aligned}$$

Thus we have arrived at a contradiction, and hence  $\rho$  must have infinite energy.

Finally, given a smooth domain  $\Omega$  as in (iv), for each  $k \gg 0$ , we have  $q_k \notin \overline{\Omega}$ . For  $m \gg 0$ , we have  $q_k \in \Omega_m$  and  $\partial \Omega \subset \Omega_m$ . Since  $\rho_k^{(m)}$  is then continuous on  $\overline{\Omega \cap \Omega_m}$ , harmonic on  $\Omega \cap \Omega_m$ , and zero on  $\partial \Omega_m$ , we also have

$$\sup_{\Omega \cap \Omega_m} \rho_k^{(m)} = \max_{\partial \Omega} \rho_k^{(m)} \quad \text{and} \quad \int_{\Omega \cap \Omega_m} |\nabla \rho_k^{(m)}|^2 dV = \int_{\partial \Omega} \rho_k^{(m)} \frac{\partial \rho_k^{(m)}}{\partial \nu} d\sigma.$$

Letting  $m \rightarrow \infty$ , and then letting  $k \rightarrow \infty$ , we get the required properties of  $\rho$  on  $\Omega$ .  $\square$

We will also use the following analogue of a theorem of Sullivan (see [Sul] and Theorem 2.1 of [NR1]):

**Lemma 4.2.** *Let  $(M, g)$  be a connected noncompact oriented complete Riemannian manifold,  $E$  an end of  $M$ , and  $h$  a positive  $C^\infty$  function on  $M$ . Assume that:*

- (i) *There exist positive constants  $K$ ,  $R_0$ , and  $\delta$  such that  $\text{Ric}_g \geq -Kg$  on  $E$  and  $\text{vol}(B(x; R_0)) \geq \delta$  for every point  $x \in E$ ;*
- (ii) *The restriction  $h|_E$  is harmonic; and*
- (iii) *For some positive constant  $C$ ,  $\int_{E \cap h^{-1}([a, b])} |\nabla h|^2 dV \leq C(b - a) + C$  for all  $a$  and  $b$  with  $0 \leq a < b$ .*

*Then  $|\nabla h|$  is bounded on  $E$ , and for each point  $p \in M$ ,*

$$\int_{E \cap B(p; R)} |\nabla h|^2 dV = O(R) \text{ as } R \rightarrow \infty.$$

*Sketch of the proof.* We may fix a nonempty compact set  $A \supset \partial E$ . As in the proof of Theorem 2.1 of [NR1], setting  $\varphi \equiv |\nabla h|^2$ , we get a positive constant  $C_1$  such that for each



point  $x_0 \in E$  with  $\text{dist}(x_0, A) > R_1 \equiv 4R_0$ ,

$$\sup_{B(x_0; R_0)} \varphi \leq C_1 \int_{B(x_0; 2R_0)} \varphi dV.$$

For  $a \equiv \inf_{B(x_0; 2R_0)} h$  and  $b \equiv \sup_{B(x_0; 2R_0)} h$ , we have on the one hand,

$$\int_{B(x_0; 2R_0)} \varphi dV \leq \int_{E \cap h^{-1}([a, b])} \varphi dV \leq C(b - a) + C.$$

On the other hand,  $b - a \leq \sup_{B(x_0, R_1)} |\nabla h| R_1$ . Combining the above, we see that if  $C_2 > 1$  is a sufficiently large positive constant that is, in particular, greater than the supremum of  $|\nabla h|$  on the  $2R_1$ -neighborhood of  $A$ , then for each point  $x_0 \in E$  at which  $|\nabla h(x_0)| > C_2$ , we have

$$\sup_{B(x_0; R_0)} |\nabla h|^2 < C_2 \sup_{B(x_0; R_1)} |\nabla h|.$$

Fixing constants  $C_3 > C_2$  and  $\epsilon > 0$  so that  $C_3^{1-\epsilon} > C_2$ , we see that if  $|\nabla h(x_0)| > C_3$ , then there exists a point  $x_1 \in B(x_0; R_1)$  such that

$$(1 + \epsilon) \log |\nabla h(x_0)| \leq \log |\nabla h(x_1)|.$$

Assuming now that  $|\nabla h|$  is unbounded on  $E$ , we will reason to a contradiction. Fixing a point  $x_0 \in E$  at which  $|\nabla h(x_0)| > C_3$  and applying the above inequality inductively, we get a sequence  $\{x_m\}$  in  $E$  such that  $\text{dist}(x_m, x_{m-1}) < R_1$  and

$$(1 + \epsilon) \log |\nabla h(x_{m-1})| \leq \log |\nabla h(x_m)|$$

for  $m = 1, 2, 3, \dots$ ; that is,  $\{|\nabla h(x_m)|\}$  has super-exponential growth. However, the local version of Yau's Harnack inequality (see [ChY]) provides a constant  $C_4 > 0$  such that

$$|\nabla h(x)| \leq C_4 h(x) \quad \text{and} \quad h(x) \leq C_4 h(p)$$

for all points  $x, p \in M$  with  $\text{dist}(p, A) > 2R_1$  and  $\text{dist}(x, p) < R_1$ , so  $\{|\nabla h(x_m)|\}$  has at most exponential growth. Thus we have arrived at a contradiction, and hence  $|\nabla h|$  must be bounded on  $E$ .

Finally, by redefining  $h$  outside a neighborhood of  $\overline{E}$ , we may assume without loss of generality that  $|\nabla h|$  is bounded on  $M$ . Fixing a point  $p \in M$ , we see that for  $R > 0$ ,  $a \equiv \inf_{B(p; R)} h$ , and  $b \equiv \sup_{B(p; R)} h$ , we have

$$\int_{E \cap B(p; R)} |\nabla h|^2 dV \leq \int_{E \cap h^{-1}([a, b])} |\nabla h|^2 dV \leq C(b - a) + C \leq C \cdot \sup |\nabla h| \cdot 2R + C.$$

Therefore,

$$\int_{E \cap B(p; R)} |\nabla h|^2 dV = O(R) \text{ as } R \rightarrow \infty.$$

□

Applying the above in the Kähler setting, we get the following:

**Proposition 4.3.** *Let  $(X, g)$  be a connected noncompact complete Kähler manifold, let  $E$  be an irregular hyperbolic end along which  $X$  has bounded geometry of order 2 (or for which there exist positive constants  $K$ ,  $R_0$ , and  $\delta$  such that  $\text{Ric}_g \geq -Kg$  on  $E$  and  $\text{vol}(B(x; R_0)) \geq \delta$  for every point  $x \in E$ ), let  $\{q_k\}$  be an irregular sequence in  $E$ , let  $G(\cdot, \cdot)$  be the Green's function on  $X$ , and let  $\rho_k \equiv G(\cdot, q_k): X \rightarrow (0, \infty]$  for each  $k$ . Then some subsequence of  $\{\rho_k\}$  converges uniformly on compact subsets of  $X$  to a function  $\rho$ . Moreover, any such limit function  $\rho$  has the following properties:*

- (i) *The function  $\rho$  is positive and pluriharmonic;*
- (ii)  $\int_E |\nabla \rho|^2 dV = \infty > \int_{X \setminus E} |\nabla \rho|^2 dV$ ;
- (iii)  $\int_{\rho^{-1}([a, b])} |\nabla \rho|^2 dV \leq b - a$  for all  $a$  and  $b$  with  $0 \leq a < b$  (in particular,  $\rho$  is unbounded on  $E$ );
- (iv) *If  $\Omega$  is any smooth domain with compact boundary (i.e., either  $\Omega$  is an end or  $\Omega \Subset X$ ) and at most finitely many terms of the sequence  $\{q_k\}$  lie in  $\Omega$ , then*

$$\sup_{\Omega} \rho = \max_{\partial \Omega} \rho < \infty \quad \text{and} \quad \int_{\Omega} |\nabla \rho|^2 dV \leq \int_{\partial \Omega} \rho \frac{\partial \rho}{\partial \nu} d\sigma < \infty;$$

and

- (v)  $|\nabla \rho|$  is bounded.

*Proof.* By Lemma 4.1, some subsequence of  $\{\rho_k\}$  converges uniformly on compact sets, and the limit  $\rho$  of any such subsequence is positive and harmonic and satisfies (ii)–(iv). Lemma 4.2 implies that  $|\nabla \rho|$  is bounded on  $E$  and for  $p \in X$ ,  $\int_{B(p; R)} |\nabla \rho|^2 dV = O(R)$  (hence  $\int_{B(p; R)} |\nabla \rho|^2 dV = o(R^2)$ ) as  $R \rightarrow \infty$ . By an observation of Gromov [Gro2] and of Li [Li] (see Corollary 2.5 of [NR1]),  $\rho$  is pluriharmonic. □

## 5. PROOF OF THE MAIN RESULT AND SOME RELATED RESULTS

This section contains the proof of Theorem 0.1. We also consider some related results.

*Proof of Theorem 0.1.* Let  $X$  be a connected noncompact hyperbolic complete Kähler manifold with bounded geometry of order two, and assume that  $X$  has exactly one end. By the main result of [NR4], the Bochner–Hartogs dichotomy holds for  $X$  regular hyperbolic. If  $X$  is irregular hyperbolic, then Proposition 4.3 provides a (quasi-Dirichlet-finite) positive pluriharmonic function  $\rho$  on  $X$  with infinite energy and bounded gradient, and hence Lemma 3.4 gives the claim.  $\square$

The above arguments together with those appearing in [NR1], [NR5], and [NR6] give results for multi-ended complete Kähler manifolds. To see this, we first recall some terminology and facts.

**Definition 5.1.** Let  $M$  be a connected manifold. Following Geoghegan [Ge] (see also Kropholler and Roller [KroR]), for  $\Upsilon: \widetilde{M} \rightarrow M$  the universal covering of  $M$ , elements of the set

$$\lim_{\leftarrow} \pi_0[\Upsilon^{-1}(M \setminus K)],$$

where the limit is taken as  $K$  ranges over the compact subsets of  $M$  (or the compact subsets of  $M$  for which the complement  $M \setminus K$  has no relatively compact components) will be called *filtered ends*. The number of filtered ends of  $M$  will be denoted by  $\tilde{e}(M)$ .

**Lemma 5.2.** *Let  $M$  be a connected noncompact topological manifold.*

- (a) *We have  $\tilde{e}(M) \geq e(M)$ . In fact for any  $k \in \mathbb{N}$ , we have  $\tilde{e}(M) \geq k$  if and only if there exists an ends decomposition  $M \setminus K = E_1 \cup \cdots \cup E_m$  such that*

$$\sum_{j=1}^m [\pi_1(M) : \Gamma_j] \geq k,$$

*where  $\Gamma_j \equiv \text{im} [\pi_1(E_j) \rightarrow \pi_1(M)]$  for  $j = 1, \dots, m$ .*

- (b) *If  $\Upsilon: \widehat{M} \rightarrow M$  is a connected covering space,  $E$  is an end of  $\widehat{M}$ , and  $E_0 \equiv \Upsilon(E) \subsetneq M$ , then*

- (i)  $E_0$  is an end of  $M$ ;
- (ii)  $\partial E_0 = \Upsilon(\partial E) \setminus E_0$ ;
- (iii)  $\overline{E} \cap \Upsilon^{-1}(\partial E_0) = (\partial E) \setminus \Upsilon^{-1}(E_0)$ ;
- (iv) *The mapping  $\Upsilon|_{\overline{E}}: \overline{E} \rightarrow \overline{E}_0$  is proper and surjective; and*
- (v) *If  $F_0 \subset E_0 \setminus \Upsilon(\partial E)$  is an end of  $M$  and  $F \equiv E \cap \Upsilon^{-1}(F_0)$ , then  $\Upsilon|_F: F \rightarrow F_0$  is a finite covering and each connected component of  $F$  is an end of  $\widehat{M}$ .*

- (c) *If  $\Upsilon: \widehat{M} \rightarrow M$  is a connected covering space, then  $\tilde{e}(\widehat{M}) \leq \tilde{e}(M)$ , with equality holding if the covering is finite.*

*Proof.* For any nonempty domain  $U$  in  $M$ , the index of  $\text{im} [\pi_1(U) \rightarrow \pi_1(M)]$  is equal to the number of connected components of the lifting of  $U$  to the universal covering of  $M$ , so part (a) holds.

For the proof of part (b), observe that  $E_0$  is a domain in  $M$ ,  $\partial E_0 \neq \emptyset$ ,  $\Upsilon(\overline{E}) \subset \overline{E}_0$ , and therefore,  $\overline{E} \cap \Upsilon^{-1}(\partial E_0) = (\partial E) \setminus \Upsilon^{-1}(E_0)$ . Given a point  $p \in M$ , we may fix domains  $U$  and  $V$  in  $M$  such that  $p \in U \Subset V$ ,  $U \cap \partial E_0 \neq \emptyset$ , and the image of  $\pi_1(V)$  in  $\pi_1(X)$  is trivial (their existence is trivial if  $p \in \partial E_0$ , while for  $p \notin \partial E_0$ , we may take  $U$  and  $V$  to be sufficiently small connected neighborhoods of the image of an injective path from  $p$  to a point in  $\partial E_0$ ). The connected components of  $\widehat{U} \equiv \Upsilon^{-1}(U)$  then form a locally finite collection of relatively compact domains in  $\widehat{M}$ , and those components that meet  $\overline{E}$  must also meet the compact set  $\partial E$ , so only finitely many components, say  $U_1, \dots, U_m$ , meet  $\overline{E}$ . Thus for  $U_0 \equiv \bigcup_{i=1}^m U_i$ , we have  $\widehat{U} \cap \overline{E} = U_0 \cap \overline{E} \Subset \widehat{M}$ , and it follows that the restriction  $\overline{E} \rightarrow \overline{E}_0$  is a proper mapping. In particular, this is a closed mapping, and hence  $\Upsilon(\overline{E}) = \overline{E}_0$ . Furthermore, the boundary

$$\partial E_0 = \Upsilon(\overline{E}) \setminus \Upsilon(E) = \Upsilon(\partial E) \setminus E_0$$

is compact and  $\overline{E}_0$  is noncompact (by properness), so  $E_0$  must be an end of  $M$ .

Finally, if  $F_0 \subset E_0 \setminus \Upsilon(\partial E)$  is an end of  $M$ , then each connected component of  $\Upsilon^{-1}(F_0)$  that meets  $E$  must lie in  $E$ . Thus the restriction  $F \equiv E \cap \Upsilon^{-1}(F_0) \rightarrow F_0$  is a covering space. Properness then implies that this restriction is actually a finite covering,  $\partial F \subset \overline{E} \cap \Upsilon^{-1}(\partial F_0)$  is compact, and in particular, each connected component of  $F$  is an end of  $\widehat{M}$ .

For the proof of part (c), let  $\widehat{\Upsilon}: \widetilde{M} \rightarrow \widehat{M}$  be the universal covering, and let  $k \in \mathbb{N}$  with  $\tilde{e}(\widehat{M}) \geq k$ . Then there exists an ends decomposition  $\widehat{M} \setminus L = F_1 \cup \dots \cup F_n$  such that  $\widehat{\Upsilon}^{-1}(\widehat{M} \setminus L)$  has at least  $k$  connected components, and there exists an ends decomposition  $M \setminus K = E_1 \cup \dots \cup E_m$  such that  $K \supset \Upsilon(L)$ . For each  $j = 1, \dots, n$ , part (b) implies that  $\Upsilon(F_j) \not\subset K$ , and hence  $F_j$  meets, and therefore contains, some connected component of  $\Upsilon^{-1}(M \setminus K)$ . Thus, under the universal covering  $\Upsilon \circ \widehat{\Upsilon}: \widetilde{M} \rightarrow M$ , the inverse image of  $M \setminus K$  has at least  $k$  connected components, and therefore  $\tilde{e}(M) \geq k$ . Thus  $\tilde{e}(M) \geq \tilde{e}(\widehat{M})$ . Furthermore, if  $\Upsilon$  is finite covering map, then the connected components of the liftings of

the ends in any ends decomposition of  $M$  form an ends decomposition for  $\widehat{M}$ . Hence in this case we have  $\tilde{e}(\widehat{M}) \geq \tilde{e}(M)$ , and therefore we have equality.  $\square$

**Definition 5.3** (cf. Definition 2.2 of [NR5]). We will call an end  $E$  of a connected non-compact complete Hermitian manifold  $X$  *special* if  $E$  is of at least one of the following types:

(BG)  $X$  has bounded geometry of order 2 along  $E$ ;

(W) There exists a continuous plurisubharmonic function  $\varphi$  on  $X$  such that

$$\{x \in E \mid \varphi(x) < a\} \Subset X \quad \forall a \in \mathbb{R};$$

(RH)  $E$  is a hyperbolic end and the Green's function vanishes at infinity along  $E$ ; or

(SP)  $E$  is a parabolic end, the Ricci curvature of  $g$  is bounded below on  $E$ , and there exist positive constants  $R$  and  $\delta$  such that  $\text{vol}(B(x; R)) > \delta$  for all  $x \in E$ .

We will call an ends decomposition in which each of the ends is special a *special ends decomposition*.

According to [Gro1], [Li], [Gro2], [GroS], [NR1], [DelG], [NR5], and [NR6], a connected noncompact complete Kähler manifold  $X$  that admits a special ends decomposition and has at least three filtered ends admits a proper holomorphic mapping onto a Riemann surface. One goal of this section is to show that if  $X$  has an irregular hyperbolic end of type (BG), then two filtered ends suffice.

**Theorem 5.4.** *If  $X$  is a connected noncompact hyperbolic complete Kähler manifold that admits a special ends decomposition  $X \setminus K = E_1 \cup \cdots \cup E_m$  for which  $E_1$  is an irregular hyperbolic end (i.e.,  $E_1$  contains an irregular sequence for  $X$ ) of type (BG) and  $m \geq 2$ , then  $X$  admits a proper holomorphic mapping onto a Riemann surface.*

*Sketch of the proof.* Every end lying in a special end is itself special, so by the main results of [NR1] and [NR5], we may assume that  $m = e(X) = 2$ . Moreover, as in the proof of Theorem 3.4 of [NR1], we may also assume that  $E_2$  is a hyperbolic end of type (BG). Theorem 2.6 of [NR1] then provides a nonconstant bounded positive Dirichlet-finite pluriharmonic function  $\rho_1$  on  $X$ . Proposition 4.3 implies that  $X$  also admits a positive (quasi-Dirichlet-finite) pluriharmonic function  $\rho_2$  with bounded gradient and infinite energy. In particular, the holomorphic 1-forms  $\theta_1 \equiv \partial\rho_1$  and  $\theta_2 \equiv \partial\rho_2$ , are linearly independent, and

Theorems 0.1 and 0.2 of [NR6] give a proper holomorphic mapping of  $X$  onto a Riemann surface.  $\square$

**Lemma 5.5** (cf. Proposition 4.1 of [NR4]). *Let  $(X, g)$  be a connected noncompact complete Kähler manifold. If  $X$  admits a special ends decomposition and some connected covering space  $\Upsilon: \widehat{X} \rightarrow X$  admits a proper holomorphic mapping onto a Riemann surface, then  $X$  admits a proper holomorphic mapping onto a Riemann surface.*

*Proof.* The Cartan–Remmert reduction of  $\widehat{X}$  is given by a proper holomorphic mapping  $\widehat{\Phi}: \widehat{X} \rightarrow \widehat{S}$  of  $\widehat{X}$  onto a Riemann surface  $\widehat{S}$  with  $\widehat{\Phi}_* \mathcal{O}_{\widehat{X}} = \mathcal{O}_{\widehat{S}}$ . Fixing a fiber  $\widehat{Z}_0$  of  $\widehat{\Phi}$ , we may form a relatively compact connected neighborhood  $\widehat{U}_0$  of  $\widehat{Z}_0$  in  $\widehat{X}$  and a nonnegative  $\mathcal{C}^\infty$  plurisubharmonic function  $\widehat{\varphi}_0$  on  $\widehat{X} \setminus \widehat{Z}_0$  such that  $\widehat{\varphi}_0$  vanishes on  $\widehat{X} \setminus \widehat{U}_0$  and  $\widehat{\varphi}_0 \rightarrow \infty$  at  $\widehat{Z}_0$ . The image  $Z_0 \equiv \Upsilon(\widehat{Z}_0)$  is then a connected compact analytic subset of  $X$ , and the function  $\varphi_0: x \mapsto \sum_{y \in \Upsilon^{-1}(x)} \widehat{\varphi}_0(y)$  is a nonnegative  $\mathcal{C}^\infty$  plurisubharmonic function on the domain  $X \setminus Z_0$  that vanishes on the complement of the relatively compact connected neighborhood  $U_0 \equiv \Upsilon(\widehat{U}_0)$  of  $Z_0$  in  $X$  and satisfies  $\varphi_0 \rightarrow \infty$  at  $Z_0$ .

We may form a special ends decomposition  $X \setminus K = E_1 \cup \cdots \cup E_m$  with  $K \supset U_0$ , and setting  $K_0 \equiv K \setminus U_0$  and  $E_0 \equiv U_0 \setminus Z_0$ , we get an ends decomposition

$$(X \setminus Z_0) \setminus K_0 = E_0 \cup E_1 \cup \cdots \cup E_m$$

of  $X \setminus Z_0$ . By part (d) of Lemma 1.1, for  $a \gg 0$ , the set  $\{x \in X \setminus Z_0 \mid \varphi(x) < a\}$  has a connected component  $Y_0$  that contains  $X \setminus U_0$  and has the ends decomposition

$$Y_0 \setminus K_0 = E'_0 \cup \cdots \cup E'_m,$$

where  $E'_j \equiv E_j \cap Y_0$  for  $j = 0, \dots, m$ . In particular, the above is a special ends decomposition for the complete Kähler metric  $g_0 \equiv g + \mathcal{L}(-\log(a - \varphi))$  on  $Y_0$  (see, for example, [Dem]), with  $E'_0$  regular hyperbolic and of type (W). Here, for any  $\mathcal{C}^2$  function  $\psi$ ,  $\mathcal{L}(\psi)$  denotes the *Levi form* of  $\psi$ ; that is, in local holomorphic coordinates  $(z_1, \dots, z_n)$ ,

$$\mathcal{L}(\psi) = \sum_{j,k=1}^n \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} dz_j d\bar{z}_k.$$

Theorem 3.6 of [NR6] implies that there exists a nonconstant nonnegative continuous plurisubharmonic function on  $Y_0$  that vanishes on  $E'_0 \cup K_0$  and, therefore, extends to a continuous plurisubharmonic function  $\alpha$  on  $X$  that vanishes on  $K$ . Fixing a fiber  $\widehat{Z}_1$  of  $\widehat{\Phi}$  through a point at which  $\alpha \circ \Upsilon > 0$ , we see that, since  $\alpha \circ \Upsilon$  is constant on  $\widehat{Z}_1$ , the image

$Z_1 \equiv \Upsilon(\widehat{Z}_1)$  must be a connected compact analytic subset of  $X \setminus K \subset Y_0 \subset X \setminus Z_0$ . As above, we get a domain  $Y_1 \subset Y_0$ , a complete Kähler metric  $g_1$  on  $Y_1$ , and a special ends decomposition of  $(Y_1, g_1)$  with at least three ends. Therefore, by Theorem 3.4 of [NR1] (or Theorem 3.1 of [NR5]), there exists a proper holomorphic mapping  $\Phi_1: Y_1 \rightarrow S_1$  of  $Y_1$  onto a Riemann surface  $S_1$  such that  $(\Phi_1)_* \mathcal{O}_{Y_1} = \mathcal{O}_{S_1}$ . Forming the complement in  $X$  of two distinct fibers of  $\Phi_1$  and applying a construction similar to the above, we get a proper holomorphic mapping  $\Phi_2: Y_2 \rightarrow S_2$  of a domain  $Y_2 \supset X \setminus Y_1$  in  $X$  onto a Riemann surface  $S_2$  such that  $(\Phi_2)_* \mathcal{O}_{Y_2} = \mathcal{O}_{S_2}$ . The maps  $\Phi_1$  and  $\Phi_2$  now determine a proper holomorphic mapping  $\Phi$  of  $X$  onto the Riemann surface

$$S \equiv (S_1 \sqcup S_2) / [\Phi_1(x) \sim \Phi_2(x) \quad \forall x \in Y_1 \cap Y_2].$$

□

*Remarks.* 1. The authors do not know whether or not the above lemma holds in general for the base an arbitrary connected noncompact complete Kähler manifold.

2. For the base a complete Kähler manifold with bounded geometry (which is the relevant case for this paper), one may instead obtain the lemma from properness of the projection from the graph over a suitable irreducible component of the appropriate Barlet cycle space as in (Theorem 3.18 and the appendix of) [Ca].

**Theorem 5.6.** *Suppose  $X$  is a connected noncompact irregular hyperbolic complete Kähler manifold with bounded geometry of order 2 and  $e(X) = 1$ . If  $X$  admits a connected covering space  $\Upsilon: \widehat{X} \rightarrow X$  with  $H_c^1(\widehat{X}, \mathcal{O}) \neq 0$ , then  $X$  admits a proper holomorphic mapping onto a Riemann surface.*

*Proof.* Clearly,  $\widehat{X}$  has bounded geometry of order 2. If  $e(\widehat{X}) \geq 3$  or  $e(\widehat{X}) = 1$ , then  $\widehat{X}$  admits a proper holomorphic mapping onto a Riemann surface, and Lemma 5.5 provides such a mapping on  $X$ . Thus we may assume that  $e(\widehat{X}) = 2$ , and we may fix an ends decomposition  $\widehat{X} \setminus K = E_1 \cup E_2$ . By part (b) of Lemma 5.2, for  $j = 1, 2$ ,  $\Upsilon(E_j)$  is a hyperbolic end of  $X$ . It follows that  $E_j$  is a hyperbolic end of  $\widehat{X}$ , since the lifting to  $\widehat{X}$  of a negative continuous subharmonic function with supremum 0 on  $X$  is a negative continuous subharmonic function on  $\widehat{X}$  with supremum 0 along  $E_j$ .

Proposition 4.3 provides an unbounded positive pluriharmonic function  $\rho_1$  with bounded gradient and infinite energy on  $X$ , and we may set  $\hat{\rho}_1 \equiv \rho_1 \circ \Upsilon$ . Theorem 2.6 of [NR1]

provides a nonconstant bounded pluriharmonic function  $\hat{\rho}_2$  with finite energy on  $\hat{X}$ , and Theorems 0.1 and 0.2 of [NR6], applied to the holomorphic 1-forms  $\partial\hat{\rho}_1$  and  $\partial\hat{\rho}_2$ , give a proper holomorphic mapping of  $\hat{X}$  onto a Riemann surface. Lemma 5.5 then gives the required mapping on  $X$ .  $\square$

Proposition 1.2 provides some topological conditions that give nonvanishing of the first compactly supported cohomology with values in the structure sheaf. In particular, since any manifold with at least two filtered ends admits a connected covering space with at least two ends, we get the following consequence of Theorem 5.6 (one may instead apply Theorem 5.4 and Lemma 5.5):

**Corollary 5.7.** *If  $X$  is a connected noncompact irregular hyperbolic complete Kähler manifold with bounded geometry of order 2,  $e(X) = 1$ , and  $\tilde{e}(X) \geq 2$ , then  $X$  admits a proper holomorphic mapping onto a Riemann surface.*

We also get the following:

**Corollary 5.8.** *Suppose  $X$  is a connected noncompact irregular hyperbolic complete Kähler manifold with bounded geometry of order 2,  $e(X) = 1$ ,  $\Omega$  is a nonempty smooth relatively compact domain in  $X$  for which  $E \equiv X \setminus \overline{\Omega}$  is connected (i.e.,  $E$  is an end), and  $\Gamma' \equiv \text{im} [\pi_1(\overline{\Omega}) \rightarrow \pi_1(X)]$ . If either  $\partial\Omega$  is not connected, or  $\partial\Omega$  is connected but  $\pi_1(\partial\Omega)$  does not surject onto  $\Gamma'$ , then  $X$  admits a proper holomorphic mapping onto a Riemann surface.*

*Proof.* If  $\partial\Omega$  is not connected, then part (a) of Proposition 1.2 implies that  $H_c^1(X, \mathcal{O}) \neq 0$ , and hence  $X$  admits a proper holomorphic mapping onto a Riemann surface. Suppose instead that  $C \equiv \partial\Omega$  is connected, but  $\Gamma \equiv \text{im} [\pi_1(C) \rightarrow \pi_1(X)] \subsetneq \Gamma'$ . For a connected covering space  $\Upsilon: \hat{X} \rightarrow X$  with  $\Upsilon_*\pi_1(\hat{X}) = \Gamma$ ,  $\Upsilon$  maps some relatively compact connected neighborhood  $U_0$  of some connected component  $C_0$  of  $\hat{C} \equiv \Upsilon^{-1}(C)$  isomorphically onto a neighborhood  $U$  of  $C$ . By Theorem 5.6, we may assume that  $e(\hat{X}) = 1$ . The unique connected component  $\Omega_0$  of  $\hat{\Omega} \equiv \Upsilon^{-1}(\Omega)$  for which  $C_0$  is a boundary component is a smooth domain, and  $C_0 \subsetneq \partial\Omega_0$ . Moreover, each component of  $\hat{X} \setminus \overline{\Omega}_0$  must meet, and therefore contain, a component of  $\Upsilon^{-1}(E)$ , so any such component must have noncompact closure. Proposition 1.2 and Theorem 5.6 together now give the claim.  $\square$



## 6. AN IRREGULAR HYPERBOLIC EXAMPLE

Because the existence of irregular hyperbolic complete Kähler manifolds with one end and bounded geometry of order two is not completely obvious, an example is provided in this section. In fact, the following is obtained:

**Theorem 6.1.** *There exists an irregular hyperbolic connected noncompact complete Kähler manifold  $X$  with bounded geometry of all orders such that  $e(X) = 1$  and  $\dim X = 1$ .*

*Remark.* The authors do not know whether or not there exists an irregular hyperbolic connected noncompact complete Kähler manifold  $X$  with bounded geometry of order 0 for which  $H_c^1(X, \mathcal{O}) = 0$  (and hence which does not admit a proper holomorphic mapping onto a Riemann surface).

The idea of the construction is as follows. The complement of a closed disk  $D$  in  $\mathbb{C}$  is irregular hyperbolic, but it has two ends. Holomorphic attachment of a suitable sequence of tubes (i.e., annuli)  $\{T_\nu\}$ , with boundary components  $A_\nu$  and  $A'_\nu$  of  $T_\nu$  for each  $\nu$ , satisfying  $A_\nu \rightarrow \infty$  and  $A'_\nu \rightarrow p \in \partial D$ , yields an irregular hyperbolic Riemann surface with one end, and a direct construction yields a Kähler metric with bounded geometry.

**Lemma 6.2.** *Let  $\{\Delta(\zeta_\nu; R_\nu)\}_{\nu=0}^\infty$  be a locally finite sequence of disjoint disks in  $\mathbb{C}$ . Then there exists a sequence of positive numbers  $\{r_\nu\}_{\nu=1}^\infty$  such that  $r_\nu < R_\nu$  for  $\nu = 1, 2, 3, \dots$ , and*

$$b \equiv \sum_{\nu=1}^{\infty} \frac{\log [R_0^{-1} R_\nu^{-1} (|\zeta_\nu - \zeta_0| + R_0)(|\zeta_\nu - \zeta_0| + R_\nu)]}{\log [R_0^{-1} r_\nu^{-1} (|\zeta_\nu - \zeta_0| + R_0)(|\zeta_\nu - \zeta_0| - r_\nu)]} < 1.$$

*Moreover, for any such sequence  $\{r_\nu\}$ , the region  $\Omega \equiv \mathbb{C} \setminus \bigcup_{\nu=1}^\infty \overline{\Delta(\zeta_\nu; r_\nu)}$  is hyperbolic, and there exists an irregular sequence  $\{\eta_k\}$  in  $\Omega$  such that  $\eta_k \rightarrow \infty$  in  $\mathbb{C}$ .*

*Proof.* It is easy to see that the above inequality will hold for all sufficiently small positive sequences  $\{r_\nu\}$ . For each  $\nu = 1, 2, 3, \dots$ , let

$$B_\nu \equiv \log [R_0^{-1} R_\nu^{-1} (|\zeta_\nu - \zeta_0| + R_0)(|\zeta_\nu - \zeta_0| + R_\nu)],$$

let

$$C_\nu \equiv \log [R_0^{-1} r_\nu^{-1} (|\zeta_\nu - \zeta_0| + R_0)(|\zeta_\nu - \zeta_0| - r_\nu)],$$

and let  $\alpha_\nu$  be the harmonic function on  $\mathbb{C} \setminus \{\zeta_0, \zeta_\nu\}$  given by

$$z \mapsto \alpha_\nu(z) \equiv \frac{1}{C_\nu} \log \left[ \frac{|z - \zeta_0|}{|z - \zeta_\nu|} \left( \frac{|\zeta_\nu - \zeta_0| + R_0}{R_0} \right) \right].$$

Clearly,  $B_\nu > 0$ , and since  $|\zeta_\nu - \zeta_0| \geq R_\nu + R_0$ ,  $C_\nu > 0$ . At each point  $z \in \partial\Delta(\zeta_0; R_0)$ , we have

$$0 \leq \alpha_\nu(z) = \frac{1}{C_\nu} \log \left[ \frac{|\zeta_\nu - \zeta_0| + R_0}{|z - \zeta_\nu|} \right] \leq \frac{B_\nu}{C_\nu},$$

since  $(|\zeta_\nu - \zeta_0| + R_\nu)|z - \zeta_\nu| \geq R_0 R_\nu$ . At each point  $z \in \partial\Delta(\zeta_\nu; r_\nu)$ , we have

$$\alpha_\nu(z) = \frac{1}{C_\nu} \log [R_0^{-1} r_\nu^{-1} (|\zeta_\nu - \zeta_0| + R_0) |z - \zeta_0|] \geq 1;$$

while at each point  $z \in \partial\Delta(\zeta_\nu; R_\nu)$ , we have

$$\alpha_\nu(z) = \frac{1}{C_\nu} \log [R_0^{-1} R_\nu^{-1} (|\zeta_\nu - \zeta_0| + R_0) |z - \zeta_0|] \leq \frac{B_\nu}{C_\nu}.$$

Moreover,

$$0 \leq \lim_{z \rightarrow \infty} \alpha_\nu(z) = \frac{1}{C_\nu} \log [R_0^{-1} (|\zeta_\nu - \zeta_0| + R_0)] \leq \frac{B_\nu}{C_\nu}.$$

Therefore, since  $\alpha_\nu$  is harmonic, we have  $\alpha_\nu \geq 0$  on

$$\mathbb{C} \setminus [\Delta(\zeta_0; R_0) \cup \Delta(\zeta_\nu; r_\nu)] \supset \overline{\Omega} \setminus \Delta(\zeta_0; R_0),$$

and  $0 \leq \alpha_\nu \leq \frac{B_\nu}{C_\nu}$  on  $\mathbb{C} \setminus [\Delta(\zeta_0; R_0) \cup \Delta(\zeta_\nu; R_\nu)]$ . Consequently, the series  $\sum_{\nu=1}^{\infty} \alpha_\nu$  converges uniformly on compact subsets of  $\overline{\Omega} \setminus \Delta(\zeta_0; R_0)$  to a nonnegative continuous function  $\alpha$  such that  $\alpha$  is positive and harmonic on  $\Omega \setminus \overline{\Delta(\zeta_0; R_0)}$ ,  $\alpha \leq b < 1$  on the set

$$I \equiv \Omega \setminus \bigcup_{\nu=0}^{\infty} \Delta(\zeta_\nu; R_\nu) = \mathbb{C} \setminus \bigcup_{\nu=0}^{\infty} \Delta(\zeta_\nu; R_\nu),$$

and for each  $\nu = 1, 2, 3, \dots$ , we have  $0 < \alpha - \alpha_\nu < 1$  on  $\overline{\Delta(\zeta_\nu; R_\nu)} \setminus \Delta(\zeta_\nu; r_\nu)$  and  $\alpha > \alpha_\nu \geq 1$  on  $\partial\Delta(\zeta_\nu; r_\nu)$ .

Clearly,  $\Omega \supset \overline{\Delta(\zeta_0; R_0)}$  is hyperbolic, and the harmonic measure of the ideal boundary of  $\Omega$  with respect to  $\Omega \setminus \overline{\Delta(\zeta_0; R_0)}$  extends to a continuous function  $u: \overline{\Omega} \setminus \Delta(\zeta_0; R_0) \rightarrow [0, 1]$ . For each  $R > R_0$ , the continuous function  $\beta_R$  on  $\overline{\Omega} \setminus \Delta(\zeta_0; R_0)$  given by

$$z \mapsto \beta_R(z) \equiv \alpha(z) + \frac{\log(|z - \zeta_0|/R_0)}{\log(R/R_0)}$$

is harmonic on  $\Omega \setminus \overline{\Delta(\zeta_0; R_0)}$  and satisfies  $\beta_R > \alpha > 1 = u$  on  $\bigcup_{\nu=1}^{\infty} \partial\Delta(\zeta_\nu; r_\nu)$ ,  $\beta_R = \alpha \geq 0 = u$  on  $\partial\Delta(\zeta_0; R_0)$ , and  $\beta_R \geq 1 \geq u$  on  $\overline{\Omega} \cap \partial\Delta(\zeta_0; R)$ . Hence  $\beta_R \geq u$  on  $(\overline{\Omega} \setminus \Delta(\zeta_0; R_0)) \cap \overline{\Delta(\zeta_0; R)}$ . Passing to the pointwise limit as  $R \rightarrow \infty$ , we get  $\alpha \geq u$  on  $\overline{\Omega} \setminus \Delta(\zeta_0; R_0)$ . However,  $\alpha \leq b < 1$  on the set  $I \subset \Omega \setminus \Delta(\zeta_0; R_0)$ , so any sequence  $\{\eta_k\}$  in  $I$  with  $\eta_k \rightarrow \infty$  in  $\mathbb{C}$  is an irregular sequence in  $\Omega$ .  $\square$

**Lemma 6.3.** *Let  $k$  be a positive integer, and  $\rho$  a positive  $\mathcal{C}^\infty$  function on  $\mathbb{C}$  such that  $D\rho, D^2\rho, D^3\rho, \dots, D^k\rho$  are bounded. Then the complete Kähler metric  $g \equiv e^{2\rho}g_{\mathbb{C}}$  has bounded geometry of order  $k$ . In fact, the pullbacks of  $g$  under the local holomorphic charts*

$$\Psi_{z_0}: \Delta(0; 1) \rightarrow \Delta(z_0; e^{-\rho(z_0)})$$

*given by  $\Psi_{z_0}: z \mapsto e^{-\rho(z_0)}z + z_0$ , for each point  $z_0 \in \mathbb{C}$ , have the appropriate uniformly bounded derivatives.*

*Proof.* For each point  $z_0 \in \mathbb{C}$ , the pullback of the associated  $(1, 1)$ -form  $\omega_g \equiv e^{2\rho} \frac{i}{2} dz \wedge d\bar{z}$  under  $\Psi_{z_0}$  is given by

$$\Psi_{z_0}^* \omega_g = e^{2(\rho(\Psi_{z_0}) - \rho(z_0))} \frac{i}{2} dz \wedge d\bar{z}.$$

The bound on  $D\rho$  gives a Lipschitz constant  $C$  for  $\rho$ , and hence

$$e^{-2C} \leq e^{-2Ce^{-\rho(z_0)}} \leq e^{2(\rho(\Psi_{z_0}) - \rho(z_0))} \leq e^{2Ce^{-\rho(z_0)}} \leq e^{2C}.$$

A similar argument gives uniform bounds on the  $m$ th order derivatives of the functions  $\left\{ e^{2(\rho(\Psi_{z_0}) - \rho(z_0))} \right\}_{z_0 \in \mathbb{C}}$  for  $m = 1, \dots, k$ .  $\square$

*Proof of Theorem 6.1. Step 1.* Construction of a suitable irregular hyperbolic region in  $\mathbb{C}$ . Let us fix a constant  $R > 1$  and disjoint disks  $\{\Delta(\zeta_\nu; R)\}_{\nu=0}^\infty$  such that  $\zeta_0 = 0$  and  $\zeta_\nu \rightarrow \infty$  so fast that

$$\sum_{\nu=1}^\infty \frac{\log [R^{-2}(|\zeta_\nu| + R)^2]}{\log [R^{-1}e^{|\zeta_\nu|}(|\zeta_\nu| + R)(|\zeta_\nu| - e^{-|\zeta_\nu|})]} < 1.$$

In particular, by Lemma 6.2, the domain

$$\Omega_0 \equiv \mathbb{C} \setminus \bigcup_{\nu=1}^\infty \overline{\Delta(\zeta_\nu; e^{-|\zeta_\nu|})}$$

is irregular hyperbolic; in fact, there exists an irregular sequence  $\{\eta_k\}$  in  $\Omega_0$  such that  $\eta_k \rightarrow \infty$  in  $\mathbb{C}$ .

**Step 2. Construction of a bounded geometry Kähler metric on a region.** By Lemma 6.3, fixing a positive  $\mathcal{C}^\infty$  function  $\rho$  on  $\mathbb{C}$  such that  $\rho(z) = |z|$  on a neighborhood of  $\mathbb{C} \setminus \Delta(0; 1)$ , we get a complete Kähler metric  $g_0 \equiv e^{2\rho}g_{\mathbb{C}}$  with bounded geometry of all orders on  $\mathbb{C}$  and associated local holomorphic charts

$$\Psi_{z_0}: \Delta(0; 1) \rightarrow \Delta(z_0; e^{-\rho(z_0)})$$

given by  $\Psi_{z_0}: z \mapsto e^{-\rho(z_0)}z + z_0$  for each point  $z_0 \in \mathbb{C}$ . Letting  $R_0$  and  $R_1$  be constants with  $1 < R_0 < R_1 < R$ ,  $g_{\mathbb{H}}$  the standard hyperbolic metric on the upper half plane  $\mathbb{H}$ ,  $\Phi$  a Möbius

transformation with  $\Phi((\mathbb{C} \setminus \overline{\Delta(0; 1)}) \cup \{\infty\}) = \mathbb{H}$  and  $\text{Im } \Phi > 5R$  on  $(\mathbb{C} \setminus \Delta(0; R_0)) \cup \{\infty\}$ ,  $g_1 \equiv \Phi^* g_{\mathbb{H}}$ , and  $\lambda: \mathbb{C} \rightarrow [0, 1]$  a  $\mathcal{C}^\infty$  function with  $\lambda \equiv 0$  on a neighborhood of  $\overline{\Delta(0; R_0)}$  and  $\lambda \equiv 1$  on  $\mathbb{C} \setminus \Delta(0; R_1)$ , we get a complete Kähler metric

$$g_2 \equiv \lambda g_0 + (1 - \lambda) g_1$$

with bounded geometry of all orders on the region  $\mathbb{C} \setminus \overline{\Delta(0; 1)}$ . Setting  $\xi_\nu \equiv 2\nu R + i2R$  for each  $\nu = 1, 2, 3, \dots$ , we get disjoint disks  $\{\Delta(\xi_\nu; R)\}_{\nu=1}^\infty$  in  $\{z \in \mathbb{H} \mid R < \text{Im } z < 3R\}$  (and an isometric isomorphism  $\Delta(\xi_1; R) \rightarrow \Delta(\xi_\nu; R)$  in  $\mathbb{H}$  given by  $z \mapsto z + 2(\nu - 1)R$  for each  $\nu$ ). We have

$$\overline{\Delta(0; 1)} \cup \bigcup_{\nu=1}^\infty \Phi^{-1}(\overline{\Delta(\xi_\nu; R)}) \subset \Delta(0; R_0) \Subset \Delta(0; R) \Subset \Omega_0,$$

and hence we have a region

$$\Omega_1 \equiv \Omega_0 \setminus \left( \overline{\Delta(0; 1)} \cup \bigcup_{\nu=1}^\infty \Phi^{-1}(\overline{\Delta(\xi_\nu; 1)}) \right).$$

**Step 3. Construction of the Riemann surface  $X$ .** For each  $\nu = 1, 2, 3, \dots$ , let  $T_\nu$  be a copy of the annulus  $\Delta(0; 1/R, R) \equiv \{z \in \mathbb{C} \mid 1/R < |z| < R\}$ , and let

$$\Lambda_\nu: \mathbb{C} \rightarrow \mathbb{C} \quad \text{and} \quad \Upsilon_\nu: \mathbb{C}^* \rightarrow \mathbb{C}^*$$

be the biholomorphisms given by  $w \mapsto e^{-|\zeta_\nu|} w + \zeta_\nu$  and  $w \mapsto \frac{1}{w} + \xi_\nu$ , respectively. We then get a Riemann surface

$$X \equiv \left( \Omega_1 \sqcup \bigsqcup_{\nu=1}^\infty T_\nu \right) / \sim,$$

where for each  $\nu = 1, 2, 3, \dots$ , and each  $w \in T_\nu$ ,  $z \in \Delta(\zeta_\nu; e^{-|\zeta_\nu|}, R e^{-|\zeta_\nu|})$  satisfies

$$z \sim w \quad \Longleftrightarrow \quad z = \Lambda_\nu(w),$$

and  $z \in \Phi^{-1}(\Delta(\xi_\nu; 1, R))$  satisfies

$$z \sim w \quad \Longleftrightarrow \quad \Phi(z) = \Upsilon_\nu(w).$$

$X$  is hyperbolic, because for each point  $z_0 \in (\partial\Delta(0; 1)) \setminus \{\Phi^{-1}(\infty)\} \subset \partial\Omega_1$ , there exists a barrier  $\beta$  on  $\Omega_1$  at  $z_0$  and a relatively compact neighborhood  $U$  of  $z_0$  in  $\mathbb{C}$  such that  $\overline{U} \setminus \overline{\Delta(0; 1)} \subset \Omega_1$  and  $\beta$  is equal to  $-1$  on  $\Omega_1 \setminus U$ , and thus we may extend  $\beta$  to a continuous subharmonic function on  $X$  that is equal to  $-1$  on  $X \setminus (\Omega_1 \cap U)$ . Fixing a disk

$$D \Subset \Delta(0; R) \cap \Omega_1 \subset X,$$

and letting  $u: X \setminus D \rightarrow [0, 1]$  be the harmonic measure of the ideal boundary of  $X$  with respect to  $X \setminus \overline{D}$ , we see that the restriction  $u|_{\Omega_0 \setminus \Delta(0; R)}$  cannot approach 1 along the sequence  $\{\eta_k\}$ , so  $X$  must be irregular hyperbolic. It is easy to see that  $e(X) = 1$ .

**Step 4. Construction of a bounded geometry Kähler metric on  $X$ .** Let us fix a  $\mathcal{C}^\infty$  function  $\tau$  on  $\mathbb{C}$  such that  $0 \leq \tau \leq 1$ ,  $\tau \equiv 1$  on  $\mathbb{C} \setminus \Delta(0; R_0)$ , and  $\tau \equiv 0$  on  $\Delta(0; 1/R_0)$ . Then we get a Kähler metric  $g$  on  $X$  by setting  $g = g_2$  on

$$\Omega_2 \equiv \mathbb{C} \setminus \left( \overline{\Delta(0; 1)} \cup \bigcup_{\nu=1}^{\infty} \overline{\Delta(\zeta_\nu; R_0 e^{-|\zeta_\nu|})} \cup \bigcup_{\nu=1}^{\infty} \Phi^{-1}(\overline{\Delta(\xi_\nu; R_0)}) \right) \subset \Omega_1 \subset X,$$

and  $g = \tau \Lambda_\nu^* g_0 + (1 - \tau) \Upsilon_\nu^* g_{\mathbb{H}}$  on  $T_\nu \subset X$  for each  $\nu = 1, 2, 3, \dots$ .

For  $\nu = 1, 2, 3, \dots$ , on  $T_\nu$  we have  $\Lambda_\nu^* g_0 = e^{2(|e^{-|\zeta_\nu|} w + \zeta_\nu| - |\zeta_\nu|)} g_{\mathbb{C}}$  and  $\Upsilon_\nu^* g_{\mathbb{H}} = \Upsilon_1^* g_{\mathbb{H}}$  (since  $\Upsilon_\nu = \Upsilon_1 + 2(\nu - 1)R$ ). Therefore, since the functions

$$w \mapsto |e^{-|\zeta_\nu|} w + \zeta_\nu| - |\zeta_\nu| \in [-R e^{-R}, R e^{-R}] \quad \text{for } \nu = 1, 2, 3, \dots,$$

have uniformly bounded derivatives of order  $k$  on  $T_\nu = \Delta(0; 1/R, R)$  for each  $k = 0, 1, 2, \dots$ ,  $(X, g)$  has bounded geometry of all orders along  $X \setminus \Omega_3$ , where

$$\Omega_3 \equiv \mathbb{C} \setminus \left( \overline{\Delta(0; 1)} \cup \bigcup_{\nu=1}^{\infty} \overline{\Delta(\zeta_\nu; R_1 e^{-|\zeta_\nu|})} \cup \bigcup_{\nu=1}^{\infty} \Phi^{-1}(\overline{\Delta(\xi_\nu; R_1)}) \right) \subset \Omega_2 \subset \Omega_1 \subset X.$$

There exists a positive constant  $r_0$  such that for each point  $z_0 \in \Omega_3 \cap \Delta(0; R_0)$ , we have

$$B \equiv B_{g_{\mathbb{H}}}(\Phi(z_0); r_0) \subset \Phi(\Omega_2 \cap \Delta(0; R_1))$$

and  $g = g_1 = \Phi^* g_{\mathbb{H}}$  on  $\Phi^{-1}(B) \subset \Omega_2 \cap \Delta(0; R_1)$ . Thus  $(X, g)$  has bounded geometry of all orders along  $\Omega_3 \cap \Delta(0; R_0)$ , as well as along the compact set  $\overline{\Delta(0; R_0, R)} \subset \Omega_1$ .

Finally, if  $r_1$  is a constant with  $0 < r_1 < \min(1, R - R_1)$ , and  $z_0 \in \Omega_3 \setminus \Delta(0; R)$ , then

$$\Delta(z_0; r_1 e^{-\rho(z_0)}) \cap \Delta(0; R_1) = \emptyset.$$

Moreover, if  $z \in \Delta(z_0; r_1 e^{-\rho(z_0)}) \cap \overline{\Delta(\zeta_\nu; R_0 e^{-|\zeta_\nu|})}$  for some  $\nu$ , then

$$\begin{aligned} R_1 e^{-|\zeta_\nu|} &< |z_0 - \zeta_\nu| < r_1 e^{-|z_0|} + R_0 e^{-|\zeta_\nu|} \leq (r_1 e^{|\zeta_\nu - z_0|} + R_0) e^{-|\zeta_\nu|} \\ &\leq (r_1 e^{r_1 e^{-|z_0|} + R_0 e^{-|\zeta_\nu|}} + R_0) e^{-|\zeta_\nu|} \leq (r_1 e^{r_1 e^{-R} + R_0 e^{-R}} + R_0) e^{-|\zeta_\nu|}. \end{aligned}$$

Thus for  $r_1$  sufficiently small, we will have, for every point  $z_0 \in \Omega_3 \setminus \Delta(0; R)$ ,

$$D_{z_0} \equiv \Delta(z_0; r_1 e^{-\rho(z_0)}) \subset \Omega_2 \setminus \Delta(0; R_1),$$

and in particular,  $g = g_2 = g_0$  on  $D_{z_0}$ . The resulting family of biholomorphisms  $\Delta(0; 1) \rightarrow D_{z_0}$  given by  $z \mapsto r_1 z e^{-|z_0|} + z_0$  for each such point  $z_0$  then have the required uniform

bounds, so  $(X, g)$  has bounded geometry of all orders along  $\Omega_3 \setminus \Delta(0; R)$ , and therefore along  $X$  itself, and completeness follows.  $\square$

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